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MANNED SPACE FLIGHT DATA PROCESSING SYSTEM
(MSF/DPS - MATHEMATICAL SUBSYSTEM)

LINEAR PROGRAMMING PRIMER

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PREFACE

This manual was written for the uninitiated. The reader with merely a knowledge of high school mathematics should be able to obtai a solution to--as well as additional information about--a linear programming problem after studying this manual. Moreover, he should be able to obtain such a solution by employing either hand calculatio or the UNIVAC 1108.

The manual is divided into three parts. Part II explains how to obtain a solution to a linear programming problem by using hand calculation. Part I provides the mathematical background necessary to understand Part II. Part III describes the technical aspects of finding a solution via the UNIVAC 1108. An interpretation of the additional information available from this computer is also given in the third part.

The reader with a knowledge of elementary linear algebra may therefore wish to begin with Part II; while the reader who wishes merely to know "how to punch and arrange his cards" may refer to Part III.

This manual is not mathematically "pure" in that proofs are excluded. However, whenever a statement is made for which a proof is required, proper reference is given for the interested reader.

This manual was written during the summer of 1970 by Patrick W. Burke, an intern assigned to NASA Headquarters in Washington, D. C. as an Administrative Specialist The author received his B.S. in Education from Duquesne University and his M.A. in Mathematics from the University of Pittsburgh. The manual was prepared under the auspices of Richard C. Tuey, Information Systems, Computer Management, Office of Manned Space Flight.

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PART I: MATHEMATICAL BACKGROUND

CHAPTER 1: MATRICES

1.1 Definitions and Notation

An $m \times n$ (read m by n) matrix is an array of elements arranged in m rows and n columns. For our purposes, the matrix elements will be real numbers.

It is convenient to have a method of designating a matrix element by its position. Hence, by a_{ij} we shall mean the element in the "i^{th}" row and the "j^{th}" column of matrix A.

Thus if A is the 3×4 matrix:

$$\begin{pmatrix} 2 & 3 & 6 & 1 \\ 0 & 7 & -2 & 11 \\ -32 & 8 & 19 & -4 \end{pmatrix},$$

$a_{11}=2$, $a_{23}=-2$, $a_{12}=3$, $a_{32}=8$, $a_{24}=11$, etc. Note that $a_{23} \neq a_{32}$ where by "$\neq$" we mean "does not equal."

We say that the <u>order</u> of the above matrix is (3,4) since there are three rows and four columns.

A <u>submatrix</u> of a given matrix is a matrix obtained from the original by deleting any number of rows and/or columns. A few submatrices of matrix A above are:

$$\begin{pmatrix} 3 & 6 & 1 \\ 7 & -2 & 11 \\ 8 & 19 & -4 \end{pmatrix}; \quad \begin{pmatrix} 2 & 6 & 1 \\ 0 & -2 & 11 \\ -32 & 19 & -4 \end{pmatrix}; \quad \begin{pmatrix} 0 & -2 \\ -32 & 19 \end{pmatrix}; \quad \begin{pmatrix} 0 & 7 & -2 & 11 \end{pmatrix};$$

$$\begin{pmatrix} 6 \\ -2 \\ 19 \end{pmatrix}; \quad \begin{pmatrix} 7 \\ 8 \end{pmatrix}; \quad (6)$$

Equality of matrices. Let A and B both be m x n matrices (i.e., they have the same order.) If the element in the i-th row and j-th column of A is equal to the element in the i-th row and j-th column of B for each element, we say that matrix A is equal to matrix B.

Symbolically, A=B if and only if $a_{ij} = b_{ij}$ for all i = 1,2,3,...,m and for all j=1,2,3,...,m.

For real numbers a and b, by "a<b", we mean that real number a is less than real number b. By "$a \leq b$", we mean that a is less than or equal to b. That is, "$a \leq b$" is true if either a<b or a=b.

Hence it is true that:

$$2 \leq 3$$
$$-7 \leq 4$$
$$2 \leq 3$$
$$5 \leq 5 \quad \text{although "} 5 < 5 \text{" is not true.}$$

If the element in the i-th row and j-th column of matrix A is less than or equal to the element in the i-th row and j-th column of matrix B for each pair of corresponding elements, we say that matrix A is less than or equal to matrix B.

Symbolically, $A \leq B$ if and only if $a_{ij} \leq b_{ij}$ for all i=1,2,...,m and for all j=1,2,...,n.

With the above definitions for comparing matrices, we have:

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 6 & 7 & -4 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 \\ 6 & 7 & -4 \end{pmatrix} = B$$

$$A = \begin{pmatrix} 1 & 2 & 8 \\ 6 & 7 & -4 \end{pmatrix} \neq \begin{pmatrix} 1 & 2 & 3 \\ 6 & 7 & -4 \end{pmatrix} = B$$

since $a_{13}=8\neq 3=b_{13}$.

$$A = \begin{pmatrix} 1 & 4 \\ 0 & 2 \\ 1 & 7 \end{pmatrix} \neq \begin{pmatrix} 1 & 7 \\ 0 & 2 \\ 1 & 4 \end{pmatrix} = B$$

since $a_{12}=4\neq 7=b_{12}$ and $a_{32}=7\neq 4=b_{32}$.

$$A = \begin{pmatrix} 1 & -3 \\ 4 & 2 \end{pmatrix} \leq \begin{pmatrix} 5 & 0 \\ 4 & 6 \end{pmatrix}$$

since $1\leq 5$, $-3\leq 0$, $4\leq 4$, and $2\leq 6$.

$$A = \begin{pmatrix} 1 & -3 \\ 4 & 2 \end{pmatrix} \not\leq \begin{pmatrix} 5 & 0 \\ 4 & 1 \end{pmatrix}$$

since $2\not\leq 1$.

$$A = \begin{pmatrix} 1 & -3 \\ 4 & 2 \end{pmatrix} \not\leq \begin{pmatrix} 100 & 20 & 17 \\ 84 & 33 & 10 \end{pmatrix} = B$$

since A is 2 x 2 and B is 2 x 3 and, hence, they cannot be compared.

$$A = (1, -7, 2, 5) \leq (8, -3, 2, 5) = B$$

By "$a\neq b$", we mean that it is not true that $a=b$, and by "$a\not\leq b$," that it is not true that $a\leq b$.

1.2 Elementary Row Operations

Let A be the m x n matrix:

$$\begin{matrix} a_{11} & a_{12} & a_{13} \cdots\cdots a_{1n} \\ a_{21} & a_{22} & a_{23} \cdots\cdots a_{2n} \\ a_{31} & a_{32} & a_{33} \cdots\cdots a_{3n} \\ \vdots & \vdots & \vdots \qquad\quad \vdots \\ a_{m1} & a_{m2} & a_{m3} \cdots\cdots a_{mn} \end{matrix}$$

There are three types of <u>elementary row operations</u> which we may perform on matrix A <u>to obtain a new matrix</u>. We may:

(1) Multiply any row by a real number other than zero.

(2) Interchange any two rows.

(3) Replace any row by itself plus a multiple of some other row.

(Note that when we speak of multiplying a row by a real number, we mean that each element in this row is multiplied by the given number.)

Let $A = \begin{pmatrix} 1 & 6 & -7 & 0 \\ 3 & 18 & -6 & 3 \\ 0 & 2 & 8 & 1 \end{pmatrix}$

$B = \begin{pmatrix} 1 & 6 & -7 & 0 \\ 1 & 6 & -2 & 1 \\ 0 & 2 & 8 & 1 \end{pmatrix}$

$C = \begin{pmatrix} 1 & 6 & -7 & 0 \\ 0 & 2 & 8 & 1 \\ 1 & 6 & -2 & 1 \end{pmatrix}$

$$D = \begin{pmatrix} 1 & 6 & -7 & 0 \\ 0 & 2 & 8 & 1 \\ 1 & -2 & -34 & -3 \end{pmatrix}$$

Matrix B is obtained from matrix A by performing an elementary row operation of type (1) on matrix A (i.e., we multiplied the second row of matrix A by 1/3).

Matrix C is obtained from matrix B by performing an elementary row operation of type (2) on matrix B (i.e., we exchanged row 2 and row 3.) Matrix D was obtained from matrix C by performing an elementary row operation of type (3) on matrix C (i.e., we replaced the third row in matrix C by itself plus (-4) times the second row).

If A and B are two m x n matrices such that matrix B is obtained from matrix A through a series of elementary row operations, we say that <u>A is row equivalent to B</u> and we write $A \sim B$. Hence, from our above example, we have $A \sim D$.

It can be shown rather easily that if $A \sim B$, then $B \sim A$. That is, if we can start with matrix A and obtain matrix B through a series of elementary row operations, then we can also start with matrix B and obtain matrix A through another series of elementary row operations. The discussion is excluded here since it does not relate to our objective. The interested reader is referred to Cullen, page 26.

A matrix is said to be in <u>row reduced echelon form</u> if it satisfies the following four conditions:

(1) The first nonzero element in each row is "1".

(2) In any column containing the first nonzero element of some row, that element is the only nonzero element in that column.

(3) The zero rows--if any--come last.

(4) When the leading "1's" in the nonzero rows are connected by a broken line, that line slopes down and to the right.

The following matrices are in row reduced echelon form:

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}; \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}; \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 4 & 5 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 9 & 0 & 6 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 6 & 2 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 3 & 0 & 8 \end{pmatrix};$$

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}; \begin{pmatrix} 1 & 0 & 0 & 3 & 2 & 1 \\ 0 & 1 & 0 & 17 & 8 & 0 \\ 0 & 0 & 1 & 5 & 6 & 7 \end{pmatrix}$$

In particular, we shall see matrices which resemble the last example above again in Part II. The following matrices are not in row reduced echelon form:

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix}; \begin{pmatrix} 1 & 5 & 0 & 1 & 0 \\ 0 & 1 & 0 & 3 & 2 \\ 0 & 0 & 1 & 0 & 0 \end{pmatrix}; \begin{pmatrix} 0 & 1 & 0 & 2 \\ 1 & 0 & 0 & 3 \\ 0 & 0 & 1 & 5 \end{pmatrix}$$

The point to be made here is that any matrix can be put in a unique row reduced echelon form through a series of elementary row operations (Cullen, page 60). That is, for any matrix, A, there exists a unique matrix, B, (where B is in row reduced echelon form) such that $A \sim B$.

Examples:

(1) Put the matrix $\begin{pmatrix} 4 & -10 & -2 & 5 \\ 1 & -2 & 1 & 1 \\ -3 & 3 & -12 & 7 \end{pmatrix}$ in row reduced echelon form.

$$\begin{pmatrix} 4 & -10 & -2 & 5 \\ 1 & -2 & 1 & 1 \\ -3 & 3 & -12 & 7 \end{pmatrix} \sim \begin{pmatrix} 1 & -2 & 1 & 1 \\ 4 & -10 & -2 & 5 \\ -3 & 3 & -12 & 7 \end{pmatrix}$$

Rearranging rows to place a "1" in a desired position.

$$\sim \begin{pmatrix} 1 & -2 & 1 & 1 \\ 0 & -2 & -6 & 1 \\ -3 & 3 & -12 & 7 \end{pmatrix}$$

Replacing the second row by itself plus (-4) times the first.

$$\sim \begin{pmatrix} 1 & -2 & 1 & 1 \\ 0 & -2 & -6 & 1 \\ 0 & -3 & -9 & 10 \end{pmatrix}$$

Replacing the third row by itself plus three times the first.

$$\sim \begin{pmatrix} 1 & -2 & 1 & 1 \\ 0 & 1 & 3 & -1/2 \\ 0 & -3 & -9 & 10 \end{pmatrix}$$

Multiplying the second row by (-1/2) to obtain another "1" in a desired position.

$$\sim \begin{pmatrix} 1 & 0 & 7 & 0 \\ 0 & 1 & 3 & -1/2 \\ 0 & -3 & -9 & 10 \end{pmatrix}$$

Replacing the first row by itself plus two times the second.

$$\sim \begin{pmatrix} 1 & 0 & 7 & 0 \\ 0 & 1 & 3 & -1/2 \\ 0 & 0 & 0 & 17/2 \end{pmatrix}$$

Replacing the third row by itself plus three times the second.

$$\sim \begin{pmatrix} 1 & 0 & 7 & 0 \\ 0 & 1 & 3 & -1/2 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Multiplying the third row by 2/17 to obtain another "1" in a desired position.

$$\sim \begin{pmatrix} 1 & 0 & 7 & 0 \\ 0 & 1 & 3 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$ Replacing the second row by itself plus (1/2) times the third.

(2) Put the matrix $\begin{pmatrix} 3 & 1 & 2 \\ 4 & 7 & 1 \\ 1 & -1 & 0 \end{pmatrix}$ in row reduced echelon form.

$$\begin{pmatrix} 3 & 1 & 2 \\ 4 & 7 & 1 \\ 1 & -1 & 0 \end{pmatrix} \sim \begin{pmatrix} 1 & -1 & 0 \\ 3 & 1 & 2 \\ 4 & 7 & 1 \end{pmatrix} \sim \begin{pmatrix} 1 & -1 & 0 \\ 0 & 4 & 2 \\ 0 & 11 & 1 \end{pmatrix} \sim \begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & 1/2 \\ 0 & 11 & 1 \end{pmatrix}$$

$$\sim \begin{pmatrix} 1 & 0 & 1/2 \\ 0 & 1 & 1/2 \\ 0 & 0 & -1/2 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1/2 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Here we have condensed our work by performing several operations in each step.

By the <u>rank of a matrix</u>, we shall mean the number on nonzero rows in its row reduced echelon form.*

Hence, the rank of the 3 x 4 matrix $\begin{pmatrix} 1 & 3 & 7 & 2 \\ -2 & 1 & 0 & 5 \\ 1 & 10 & 21 & 11 \end{pmatrix}$ is two since it can be shown that its row reduced echelon form is:

$$\begin{pmatrix} 1 & 0 & 1 & -13/7 \\ 0 & 1 & 2 & 9/7 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

* In most texts, this is stated as the result of another definition of matrix rank.

1.3 Determinant of a Square Matrix

In the expression: (2)(4)(8)+(6)(9)-(20)(17)(-8)(5), there are three terms. The first term contains three factors; the second term contains two factors; and the third term contains four factors.

A square matrix is a matrix with the same number of rows and columns (i.e., m=n).

For every square matrix, A, there exists a real number called the determinant of A (written $|A|$).

By definition, if A is the n x n square matrix

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} \cdots\cdots a_{1n} \\ a_{21} & a_{22} & a_{23} \cdots\cdots a_{2n} \\ a_{31} & a_{32} & a_{33} \cdots\cdots a_{3n} \\ \vdots & \vdots & \vdots \qquad\quad \vdots \\ a_{n1} & a_{n2} & a_{n3} \cdots\cdots a_{nn} \end{pmatrix}$$

$|A|$ = the sum of all possible terms of the form

$$a_{1i}a_{2j}a_{3k}\cdots\cdots a_{nr}$$

where each term is preceded by an appropriate sign (+ or -) which is determined by a specific rule (Hadley, page 30). That is, each term contains n factors. The only restriction is that each term must contain one and only one factor from each row and one and only one factor from each column. Thus, if A is the 4 x 4 square matrix:

$$\begin{pmatrix} 1 & 2 & 3 & 7 \\ -8 & 4 & 1 & 0 \\ 1 & 9 & 2 & 6 \\ -10 & 0 & 3 & 5 \end{pmatrix}$$

A would contain the following terms: (1)(4)(2)(5), (3)(-8)(6)(0), (2)(0)(2)(-10), etc. but would not contain (3)(4)(2)(5) since 3 and 2 are both in the same column.

If we choose the first factor of each term from the first row and the second factor from the second row, etc., each term will have one and only one factor from each row. If we then discretely select each factor so that no term has more than one factor from each column, we will have listed each term in $|A|$

Since there are four choices for the first factor of a term and three choices for the second factor (after the first has been selected) and two choices for the third factor (after the first two have been selected) and only one choice for the fourth factor (after the first three have been chosen), there will be $4! = (4)(3)(2)(1) = 24$ terms in $|A|$.

The reader may be familiar with certain "tricks" for evaluating the determinant of a 2 x 2 or a 3 x 3 matrix.

For example, if $C=\begin{pmatrix} 1 & 2 \\ -8 & 4 \end{pmatrix}$, and $B=\begin{pmatrix} 1 & 2 & 3 \\ -8 & 4 & 1 \\ 1 & 9 & 2 \end{pmatrix}$,

$$|C| = \begin{vmatrix} 1 & 2 \\ -8 & 4 \end{vmatrix} = (1)(4)-(-8)(2)=20$$

$$|B| = \left|\begin{matrix} 1 & 2 & 3 & 1 & 2 \\ -8 & 4 & 1 & -8 & 4 \\ 1 & 9 & 2 & 1 & 9 \end{matrix}\right.$$

$$= (1)(4)(2)+(2)(1)(1)+(3)(-8)(9)-(1)(4)(3)-(9)(1)(1)-(2)(-8)(2)$$

$$= -195$$

It should be kept in mind, however, that these "tricks" for evaluating a determinant are a result of the definition (page 9). The problem arises when one tries to extend these techniques to evaluate a determinant of matrix A where A is the matrix above.

$$\text{Example:}\quad A = \begin{pmatrix} 1 & 2 & 3 & 7 \\ -8 & 4 & 1 & 0 \\ 1 & 9 & 2 & 6 \\ -10 & 0 & 3 & 5 \end{pmatrix}$$

Copying the first three columns to the right of the determinant and attempting to use our previous tricks, we would have:

$$|A| = \begin{array}{|ccccccc} 1 & 2 & 3 & 7 & 1 & 2 & 3 \\ -8 & 4 & 1 & 0 & -8 & 4 & 1 \\ 1 & 9 & 2 & 6 & 1 & 9 & 2 \\ -10 & 0 & 3 & 5 & -10 & 0 & 3 \end{array}$$

$$= (1)(4)(2)(5)+(2)(1)(6)(-10)+(3)(0)(1)(0)+(7)(-8)(9)(3)$$
$$-(-10)(9)(1)(7)-(0)(2)(0)(1)-(3)(6)(-8)(2)-(5)(1)(4)(3)$$

We see that we obtain only eight of the twenty-four terms in the sum by this technique. In particular, the term (3)(-8)(6)(0) (which we previously agreed to include) is missing--along with the other fifteen terms which should be included in the sum according to the definition of a determinant. We therefore conclude that the "tricks" cited above which work when evaluating a 2 x 2 or a 3 x 3 determinant cannot be used when evaluating a determinant of order greater than three.

Fortunately, however, we do not have to resort to the definition every time we evaluate a determinant of order four or more.

A scheme is available which enables us to evaluate the determinant of any square matrix A.

Let A be an n x n matrix. Recall that a_{ij} is the element in the i-th row and the j-th column of A. Denote by A_{ij} the submatrix of A which is obtained by deleting the i-th row and j-th column. Hence A_{ij} is an n-1 x n-1 matrix. A_{ij} is called the minor of element a_{ij}. $(-1)^{i+j}|A_{ij}|$ is a real number and is called the cofactor of element a_{ij}.

For a nonnegative integer (whole number) p,
Recall that $(-1)^p = \begin{cases} 1, \text{ if p is even} \\ -1, \text{ if p is odd} \end{cases}$

It can be shown (Cullen, page 69) that if we choose a row of matrix A (i.e., fix i), then $|A|$ can be evaluated as follows:

$$|A| = \sum_{j=1}^{n} (-1)^{i+j} a_{ij} |A_{ij}| = (-1)^{i+1} a_{i1} |A_{i1}| + (-1)^{i+2} a_{i2} |A_{i2}| + \ldots$$
$$\ldots + (-1)^{i+n} a_{in} |A_{in}| \quad \text{for any } i=1,2,\ldots$$

Likewise, we can choose any column of matrix A (i.e. fix j). We would then have:

$$|A| = \sum_{i=1}^{n} (-1)^{i+j} a_{ij} |A_{ij}| \quad \text{for any } j=1,2,\ldots,n$$

Formally stated, we can evaluate the determinant of matrix A through expansion by minors of row i or through expansion by minors of column j.

Example: Let A be the 4 x 4 matrix cited above (page 11). Let us evaluate $|A|$ through expansion by minors of the second column.

$$|A| = (-1)^{1+2}(2)|A_{12}| + (-1)^{2+2}(4)|A_{22}| + (-1)^{3+2}(9)|A_{32}| + (-1)^{4+2}(0)|A_{42}$$

Here $|A_{12}|$, $|A_{22}|$, and $|A_{32}|$ are 3 x 3 determinants and can be evaluated by the trick discussed before which is valid for 2 x 2 and 3 x 3 determinants. If we choose, however, we may evaluate the 3 x 3 determinants $|A_{12}|$, $|A_{22}|$, and $|A_{32}|$ through expansion by minors of any of their three rows or columns. Note that we need not bother evaluating $|A_{42}|$ since $|a_{42}|=0$. That is, $(-1)^{4+2}(0)\ |A_{42}| = 0$ no matter what $|A_{42}|$ is.

It is therefore advisable to evaluate a determinant through minors of the row or column which contains the most zeros.

Returning to the above expansion, since

$|A_{12}| = -1$, $|A_{22}| = -42$, and $|A_{32}| = 27$, we have:

$|A| = -(2)(-1)+(4)(-42)-(9)(27)+0 = -325$

1.4 Addition and Multiplication of Matrices

Let A and B be n x m matrices

Example: $$A = \begin{pmatrix} a_{11} & a_{12} \cdots\cdots & a_{1n} \\ a_{21} & a_{22} \cdots\cdots & a_{2n} \\ \vdots & \vdots & \vdots \\ a_{m1} & a_{m2} \cdots\cdots & a_{mn} \end{pmatrix}$$

$$B = \begin{pmatrix} b_{11} & b_{12} \cdots\cdots & b_{1n} \\ b_{12} & b_{22} \cdots\cdots & b_{2n} \\ \vdots & \vdots & \vdots \\ b_{m1} & b_{m2} \cdots\cdots & b_{mn} \end{pmatrix}$$

We define the sum, A + B, to be the m x n matrix:

$$D = \begin{pmatrix} d_{11} & d_{12} \cdots\cdots & d_{1n} \\ d_{21} & d_{22} \cdots\cdots & d_{2n} \\ \vdots & \vdots & \vdots \\ d_{m1} & d_{m2} \cdots\cdots & d_{mn} \end{pmatrix}$$

where $d_{ij} = a_{ij} + b_{ij}$. That is, to add two matrices, we merely add corresponding elements.

Example: $$\begin{pmatrix} 1 & 0 & 7 \\ 2 & -3 & 8 \end{pmatrix} + \begin{pmatrix} 9 & 11 & 0 \\ -6 & 1 & 2 \end{pmatrix} = \begin{pmatrix} 1+9 & 0+11 & 7+0 \\ 2+(-6) & (-3)+1 & 8+2 \end{pmatrix}$$

$$= \begin{pmatrix} 10 & 11 & 7 \\ -4 & -2 & 10 \end{pmatrix}$$

Note that we only define the addition of two matrices which have the same order. Hence $\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} + \begin{pmatrix} 5 & 6 \\ 7 & 8 \\ 9 & 10 \end{pmatrix}$ is meaningless.

Now let A be an $\underline{m \times n}$ matrix and B an $\underline{n \times q}$ matrix. That is, the number of columns in A is equal to the number of rows in B.

Example: $A = \begin{pmatrix} a_{11} & a_{12} \cdots\cdots & a_{1n} \\ a_{21} & a_{22} \cdots\cdots & a_{2n} \\ \vdots & \vdots & \vdots \\ a_{m1} & a_{m2} \cdots\cdots & a_{mn} \end{pmatrix}$; $B = \begin{pmatrix} b_{11} & b_{12} \cdots\cdots & b_{1q} \\ b_{21} & b_{22} \cdots\cdots & b_{2q} \\ \vdots & \vdots & \vdots \\ b_{n1} & b_{n2} \cdots\cdots & b_{nq} \end{pmatrix}$

We define the product, AB, to be the $\underline{m \times q}$ matrix:

$$C = \begin{pmatrix} c_{11} & c_{12} \cdots\cdots & c_{1q} \\ c_{21} & c_{22} \cdots\cdots & c_{2q} \\ \vdots & \vdots & \vdots \\ c_{m1} & c_{m2} \cdots\cdots & c_{mq} \end{pmatrix}$$

where $d_{ij} = \sum_{k=1}^{n} a_{ik}b_{kj} = a_{i1}b_{1j} + a_{i2}b_{2j} + \cdots + a_{in}b_{nj}$. That is, to obtain the element in the i-th row and j-th column of the product matrix, we take the i-th row of A and the j-th column of B and add corresponding elements.

Examples:

(1) $\underset{\underline{2 \times 2}}{\begin{pmatrix} 1 & 0 & -2 \\ 3 & 4 & 1 \end{pmatrix}} \underset{\underline{3 \times 4}}{\begin{pmatrix} 2 & 1 & 0 & 2 \\ 1 & -3 & 4 & 6 \\ 0 & 5 & 2 & 1 \end{pmatrix}}$

$$= \begin{pmatrix} (1)(2)+(0)(1)+(-2)(0) & (1)(1)+(0)(-3)+(-2)(5) & (1)(0)+(0)(4)+(-2)(2) & (1)(2)+(0)(6)+(-2)(1) \\ (3)(2)+(4)(1)+(1)(0) & (3)(1)+(4)(-3)+(1)(5) & (3)(0)+(4)(4)+(1)(2) & (3)(2)+(4)(6)+(1)(1) \end{pmatrix}$$

$$= \underset{\underline{2 \times 4}}{\begin{pmatrix} 2 & -9 & -4 & 0 \\ 10 & -4 & 18 & 31 \end{pmatrix}}$$

(2) $\underset{\underline{2 \times 4}}{\begin{pmatrix} 4 & 6 & -2 & 1 \\ 1 & 0 & 3 & 5 \end{pmatrix}} \underset{\underline{4 \times 1}}{\begin{pmatrix} 2 \\ 1 \\ 3 \\ -1 \end{pmatrix}}$

$$= \begin{pmatrix} (4)(2)+(6)(1)+(-2)(3)+(1)(-1) \\ (1)(2)+(0)(1)+(3)(3)+(5)(-1) \end{pmatrix} = \underset{\underline{2 \times 1}}{\begin{pmatrix} 7 \\ 6 \end{pmatrix}}$$

(3) $$\underset{1 \times 4}{\begin{pmatrix} 8 & 2 & -4 & 3 \end{pmatrix}} \underset{4 \times 1}{\begin{pmatrix} 1 \\ 0 \\ 2 \\ 5 \end{pmatrix}} = \Big((8)(1)+(2)(0)+(-4)(2)+(3)(5) \Big) = \underset{1 \times 1}{(15)}$$

Note that we define matrix multiplication only if the number of columns in the left matrix is equal to the number of rows in the right matrix. Hence if A and B are the matrices of example (1), BA is meaningless since there are four columns in the left matrix, B, but two rows in the right matrix, A.

Hence we see that AB $\neq$ BA since AB was the 2×4 product matrix of example (1), while BA is not even defined.

It may happen that both AB and BA are defined. This will be the case if A and B are both square matrices of the same order.

Example:

$$\text{Let } A = \begin{pmatrix} 2 & 1 & 3 \\ 0 & -2 & 5 \\ 1 & 3 & 0 \end{pmatrix}; \qquad B = \begin{pmatrix} 1 & 5 & 0 \\ 2 & 3 & 1 \\ 0 & -1 & 4 \end{pmatrix}$$

Since A is 3×3 and B is 3×3, AB is 3×3 and for the same reason, BA is 3×3. We have:

$$AB = \begin{pmatrix} (2)(1)+(1)(2)+(3)(0) & (2)(5)+(1)(3)+(3)(-1) & (2)(0)+(1)(1)+(3)(4) \\ (0)(1)+(-2)(2)+(5)(0) & (0)(5)+(-2)(3)+(5)(-1) & (0)(0)+(-2)(1)+(5)(4) \\ (1)(1)+(3)(2)+(0)(0) & (1)(5)+(3)(3)+(0)(-1) & (1)(0)+(3)(1)+(0)(4) \end{pmatrix}$$

$$= \begin{pmatrix} 4 & 10 & 13 \\ -4 & -11 & 18 \\ 7 & 14 & 3 \end{pmatrix}$$

$$BA = \begin{pmatrix} (1)(2)+(5)(0)+(0)(1) & (1)(1)+(5)(-2)+(0)(3) & (1)(3)+(5)(5)+(0)(0) \\ (2)(2)+(3)(0)+(1)(1) & (2)(1)+(3)(-2)+(1)(3) & (2)(3)+(3)(5)+(1)(0) \\ (0)(2)+(-1)(0)+(4)(1) & (0)(1)+(-1)(-2)+(4)(3) & (0)(3)+(-1)(5)+(4)(0) \end{pmatrix}$$

$$= \begin{pmatrix} 2 & -9 & 28 \\ 5 & -1 & 21 \\ 4 & 14 & -5 \end{pmatrix}$$

Note that in this case also, we have $AB \neq BA$. Because of this, we

A: say that "matrix multiplication is <u>not</u> commutative".

B: However, matrix multiplication is <u>associative</u>. By this we mean that if A is an m x n matrix; B is an n x q matrix; and C is a q x r matrix, then

$$(AB)C = A(BC)$$

That is, we will obtain equal matrices by either of the following procedures:

(1) Multiply the m x n matrix A by the n x q matrix B on the right to obtain the m x q matrix AB. Then multiply the matrix AB on the right by the q x r matrix C to obtain the m x r matrix (AB)C.

(2) Multiply the n x q matrix B on the right by the q x r matrix C to obtain the n x r matrix BC. Then multiply the matrix BC on the left by the m x n matric A to obtain the m x r matrix A(BC).

C: We also have that A=B implies AC=BC and CA=CB for any matrices C and D for which the indicated multiplication is defined.

1.5 The Inverse of a Square Matrix

A <u>square</u> matrix of the form

$$\begin{matrix} 1 & 0 & 0 \ldots\ldots\ldots 0 \\ 0 & 1 & 0 \ldots\ldots\ldots 0 \\ 0 & 0 & 1 \ldots\ldots\ldots 0 \\ & & \\ 0 & 0 & 0 \ldots\ldots\ldots 1 \end{matrix}$$

(i.e., All elements on the diagonal are "1" while all other elements are "0".) is called an <u>identity matrix</u> and denoted by I (or by I_n if we wish to specify that it is n x n).

Note that if A is any square n x n matrix and I is the n x n identity matrix, then AI = IA = A.

Example: Let A be the 3 x 3 matrix $\begin{pmatrix} 4 & 1 & -2 \\ 3 & -6 & 5 \\ 0 & 8 & 7 \end{pmatrix}$ then

$$AI = \begin{pmatrix} 4 & 1 & -2 \\ 3 & -6 & 5 \\ 0 & 8 & 7 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} (4)(1)+(1)(0)+(-2)(0) & (4)(0)+(1)(1)+(-2)(0) & (4)(0)+(1)(0)+(-2)(1) \\ (3)(1)+(-6)(0)+(5)(0) & (3)(0)+(-6)(1)+(5)(0) & (3)(0)+(-6)(0)+(5)(1) \\ (0)(1)+(8)(0)+(7)(0) & (0)(0)+(8)(1)+(7)(0) & (0)(0)+(8)(0)+(7)(1) \end{pmatrix}$$

$$= \begin{pmatrix} 4 & 1 & -2 \\ 3 & -6 & 5 \\ 0 & 8 & 7 \end{pmatrix}$$

Likewise

$$IA = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 4 & 1 & -2 \\ 3 & -6 & 5 \\ 0 & 8 & 7 \end{pmatrix} = \begin{pmatrix} (1)(4)+(0)(3)+(0)(0) & (1)(1)+(0)(-6)+(0)(8) & (1)(-2)+(0)(5)+(0)(7) \\ (0)(4)+(1)(3)+(0)(0) & (0)(1)+(1)(-6)+(0)(8) & (0)(-2)+(1)(5)+(0)(7) \\ (0)(4)+(0)(3)+(1)(0) & (0)(1)+(0)(-6)+(1)(8) & (0)(-2)+(0)(5)+(1)(7) \end{pmatrix}$$

$$= \begin{pmatrix} 4 & 1 & -2 \\ 3 & -6 & 5 \\ 0 & 8 & 7 \end{pmatrix}$$

Given a square matrix A , two questions arise:

(1) Is there a square matrix B (of the same order) such that AB = I?

(2) If such a matrix exists, how do we calculate its elements?

If such a matrix, B, exists, we say that A is nonsingular and call B the inverse of A. We designate this by writing $B = A^{-1}$.

A: It can be shown that if $B = A^{-1}$ exists, then AB=BA=I

If no such B exists, we say that A is singular.

It can be shown (Cullens, page 74) that a condition which implies that A^{-1} exists and which is implied by the existence of A^{-1} is that $|A| \neq 0$

* * * Note that we talk about A^{-1} and $|A|$ only for square matrices A.* * *

B: If we have an n x n square matrix A for which we have decided that there exists an inverse B (i.e., we have found that $|A| \neq 0$), we proceed to contruct B as follows:

$$B = \begin{pmatrix} b_{11} & b_{12} \cdots\cdots & b_{1n} \\ b_{21} & b_{22} \cdots\cdots & b_{2n} \\ \vdots & \vdots & \vdots \\ b_{n1} & b_{n2} \cdots\cdots & b_{nn} \end{pmatrix}$$

where $b_{ij} = \dfrac{(-1)^{i+j} A_{ji}}{|A|}$. We recall that A_{ji} is the submatrix of A obtained by deleting the j-th row and the i-th column.

Hence, b_{ij}, the element in the i-th row and j-th column of the inverse of A, is the cofactor of a_{ji}, the element in the j-th row and i-th column of A, divided by the determinant of A (which is not zero).

Example:

$$\text{Let } A = \begin{pmatrix} 2 & 0 & 1 \\ -1 & 3 & 4 \\ 0 & 5 & 0 \end{pmatrix}$$

$$\text{Since } |A| = (-1)^{1+1}(2)\begin{vmatrix} 3 & 4 \\ 5 & 0 \end{vmatrix} + (-1)^{2+1}(-1)\begin{vmatrix} 0 & 1 \\ 5 & 0 \end{vmatrix} + (-1)^{3+1}(0)\begin{vmatrix} 0 & 1 \\ 3 & 4 \end{vmatrix}$$

$$= 2(0-20) + (0-5) + 0$$

$$= -45 \neq 0 \quad ,$$

we know that there exists a 3 x 3 matrix, B, such that $AB=BA=I = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$.

That is, we know that A^{-1} exists. In the above evaluation of $|A|$, we expanded by minors of the first column.

We proceed to calculate A_{ij} for i,j = 1,2,3. We have:

$$|A_{11}| = \begin{vmatrix} 3 & 4 \\ 5 & 0 \end{vmatrix} = 0-20 = -20$$

$$|A_{12}| = \begin{vmatrix} -1 & 4 \\ 0 & 0 \end{vmatrix} = 0-0 = 0$$

$$|A_{13}| = \begin{vmatrix} -1 & 3 \\ 0 & 5 \end{vmatrix} = -5-0 = -5$$

$$|A_{21}| = \begin{vmatrix} 0 & 1 \\ 5 & 0 \end{vmatrix} = 0-5 = -5$$

$$|A_{22}| = \begin{vmatrix} 2 & 1 \\ 0 & 0 \end{vmatrix} = 0-0 = 0$$

$$|A_{23}| = \begin{vmatrix} 2 & 0 \\ 0 & 5 \end{vmatrix} = 10-0 = 10$$

$$|A_{31}| = \begin{vmatrix} 0 & 1 \\ 3 & 4 \end{vmatrix} = 0-3 = -3$$

$$|A_{32}| = \begin{vmatrix} 2 & 1 \\ -1 & 4 \end{vmatrix} = 8-(-1) = 9$$

$$|A_{33}| = \begin{vmatrix} 2 & 0 \\ -1 & 3 \end{vmatrix} = 6-0 = 6$$

Hence

$$B=A^{-1}= \begin{pmatrix} \frac{(-1)^{1+1}|A_{11}|}{|A|} & \frac{(-1)^{2+1}|A_{21}|}{|A|} & \frac{(-1)^{3+1}|A_{31}|}{|A|} \\ \frac{(-1)^{1+2}|A_{12}|}{|A|} & \frac{(-1)^{2+2}|A_{22}|}{|A|} & \frac{(-1)^{3+2}|A_{32}|}{|A|} \\ \frac{(-1)^{1+3}|A_{13}|}{|A|} & \frac{(-1)^{2+3}|A_{23}|}{|A|} & \frac{(-1)^{3+3}|A_{33}|}{|A|} \end{pmatrix}$$

$$= \begin{pmatrix} \frac{(-20)}{-45} & \frac{(-1)(-5)}{-45} & \frac{(-3)}{-45} \\ \frac{(-1)(0)}{-45} & \frac{0}{-45} & \frac{(-1)(9)}{-45} \\ \frac{(-5)}{-45} & \frac{(-1)(10)}{-45} & \frac{(6)}{-45} \end{pmatrix}$$

$$= \begin{pmatrix} 4/9 & -1/9 & 1/15 \\ 0 & 0 & 1/5 \\ 1/9 & 2/9 & -2/15 \end{pmatrix}$$

Note that

$$AB=AA^{-1}=\begin{pmatrix} (2)(4/9)+(0)(0)+(1)(1/9) & (2)(-1/9)+(0)(0)+(1)(2/9) & (2)(1/15)+(0)(1/5)+(1)(-2/15) \\ (-1)(4/9)+(3)(0)+(4)(1/9) & (-1)(-1/9)+(3)(0)+(4)(2/9) & (-1)(1/15)+(3)(1/5)+(4)(-2/15) \\ (0)(4/9)+(5)(0)+(0)(1/9) & (0)(-1/9)+(5)(0)+(0)(2/9) & (0)(1/15)+(5)(1/5)+(0)(-2/15) \end{pmatrix}$$

$$=\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Likewise

$$BA=A^{-1}A=\begin{pmatrix} (4/9)(2)+(-1/9)(-1)+(1/15)(0) & (4/9)(0)+(-1/9)(3)+(1/15)(5) & (4/9)(1)+(-1/9)(4)+(1/15)(0) \\ (0)(2)+(0)(-1)+(1/5)(0) & (0)(0)+(0)(3)+(1/5)(5) & (0)(1)+(0)(4)+(1/5)(0) \\ (1/9)(2)+(2/9)(-1)+(-2/15)(0) & (1/9)(0)+(2/9)(3)+(-2/15)(5) & (1/9)(1)+(2/9)(4)+(-2/15)(0) \end{pmatrix}$$

$$=\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Needless to say, the construction of A^{-1} for a large matrix A would be a tedious task. The utilization of an electronic computer when working with a problem which entailed this task would prove invaluable.

CHAPTER 2: SYSTEMS OF SIMULTANEOUS LINEAR EQUATIONS

2.1 Definitions

A linear equation has the form:

$$a_1x_1 + a_2x_2 + a_nx_n = b$$

where a_1, a_2, a_3,........, a_n and b are constants, while x_1, x_2, x_3,.... ...., x_n are the "unknowns" or variables. By a solution to the above linear equation, we shall mean an n x 1 matrix $\begin{pmatrix} s_1 \\ s_2 \\ \vdots \\ s_n \end{pmatrix}$ such that the above equation is true if we substitute s_1 for x_1, s_2 for x_2, etc.

Example: $\begin{pmatrix} 5 \\ 2 \\ 1 \\ -6 \end{pmatrix}$ is a solution to the linear equation:

$$4x_1 + 3x_2 - 7x_3 + 2x_4 = 7$$

since $4(5) + 3(2) - 7(1) + 2(-6) = 7$. Note that $\begin{pmatrix} 3 \\ 1 \\ 0 \\ -4 \end{pmatrix}$ is also a solution.

Let us now consider the following system of simultaneous linear equations:

$$\begin{aligned} 2x_1 + x_2 + 2x_3 &= 3 \\ x_1 + 3x_2 + 2x_3 &= -1 \\ -x_1 + 2x_2 - x_3 &= -4 \end{aligned}$$

What does it mean to "solve" such a system? By a <u>solution</u> to this system, we shall mean a 3 x 1 matrix $\begin{pmatrix} s_1 \\ s_2 \\ s_3 \end{pmatrix}$ such that $\begin{pmatrix} s_1 \\ s_2 \\ s_3 \end{pmatrix}$ is a solution to <u>each</u> of the three linear equations in the system.

Hence, $\begin{pmatrix} -5 \\ -2 \\ 5 \end{pmatrix}$ is <u>not</u> a solution to the system even though it satisfies the second and third equations since it does not satisfy the first.

At this point, we do not know whether or not the above system has a solution; and if it does, whether it has one or many. That is, a system of simultaneous linear equations may have:

(1) no solutions

(2) one solution

(3) many solutions

2.2 Finding a Solution

Two systems of simultaneous linear equations are said to be <u>equivalent</u> if they have the same solutions.

Thus, when faced with finding the solution or solutions (if any exists) of a system of simultaneous linear equations, our objective will be to obtain an equivalent system (i.e., one with the same solutions as the original) but which is easier to solve.

Given the general system of simultaneous linear equations:

$$\begin{aligned}
a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \cdots\cdots + a_{1n}x_n &= b_1 \\
a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + \cdots\cdots + a_{2n}x_n &= b_2 \\
a_{31}x_1 + a_{32}x_2 + a_{33}x_3 + \cdots\cdots + a_{3n}x_n &= b_3 \\
&\vdots \\
a_{m1}x_1 + a_{m2}x_2 + a_{m3}x_3 + \cdots\cdots + a_{mn}x_n &= b_m
\end{aligned}$$

an equivalent system will be obtained if we:

(1) Interchange any two equations.

(2) Multiply both sides of any equation by a nonzero constant.

(3) Replace any equation by itself plus some multiple of any other equation.

The first two assertions are easy enough to justify; the third requires only a little more contemplation.

Let us use these three tricks repeatedly to obtain a "convenient" system of simultaneous linear equations which is equivalent to our original system on page 27.

(System #1; Original)

$$\begin{aligned}
2x_1 + x_2 + 2x_3 &= 3 \\
x_1 + 3x_2 + 2x_3 &= -1 \\
-x_1 + 2x_2 - x_3 &= -4
\end{aligned}$$

Let us first interchange the first and second equations to obtain the equivalent system:

(System #2)
$$\begin{aligned} x_1 + 3x_2 + 2x_3 &= -1 \\ 2x_1 + x_2 + 2x_3 &= 3 \\ -x_1 + 2x_2 - x_3 &= -4 \end{aligned}$$

Let us now replace the second equation by itself plus (-2) times the first equation to obtain the equivalent system:

(System #3)
$$\begin{aligned} x_1 + 3x_2 + 2x_3 &= -1 \\ - 5x_2 - 2x_3 &= 5 \\ -x_1 + 2x_2 - x_3 &= -4 \end{aligned}$$

Let us now replace the third equation in this system by itself plus one times the first equation to obtain the equivalent system:

(System #4)
$$\begin{aligned} x_1 + 3x_2 + 2x_3 &= -1 \\ - 5x_2 - 2x_3 &= 5 \\ 5x_2 + x_3 &= -5 \end{aligned}$$

Let us now multiply the second equation by (-1/5) to obtain the equivalent system:

(System #5)
$$\begin{aligned} x_1 + 3x_2 + 2x_3 &= -1 \\ x_2 + 2/5x_3 &= -1 \\ 5x_2 + x_3 &= -5 \end{aligned}$$

Let us now replace the first equation by itself plus (-3) times the second equation to obtain the equivalent system:

(System #6)
$$\begin{aligned} x_1 \qquad + 4/5x_3 &= 2 \\ x_2 + 2/5x_3 &= -1 \\ 5x_2 + x_3 &= -5 \end{aligned}$$

Let us now replace the third equation by itself plus (-5) times the second to obtain the equivalent system:

$$\text{(System \#7)}\qquad \begin{aligned} x_1 \qquad + 4/5\, x_3 &= 2 \\ x_2 + 2/5\, x_3 &= -1 \\ - \quad x_3 &= 0 \end{aligned}$$

Let us now multiply the third equation by (-1) to obtain the equivalent system:

$$\text{(System \#8)}\qquad \begin{aligned} x_1 \qquad + 4/5\, x_3 &= 2 \\ x_2 + 2/5\, x_3 &= -1 \\ x_3 &= 0 \end{aligned}$$

Let us now replace the second equation by itself plus (-2/5) times the third to obtain the equivalent system:

$$\text{(System \#9)}\qquad \begin{aligned} x_1 \qquad + 4/5\, x_3 &= 2 \\ x_2 \qquad &= -1 \\ x_3 &= 0 \end{aligned}$$

Let us now replace the first equation by itself plus (-4/5) times the third to obtain the equivalent system:

$$\text{(System \#10)}\qquad \begin{aligned} x_1 &= 2 \\ x_2 &= -1 \\ x_3 &= 0 \end{aligned}$$

We now have a system of simultaneous linear equations for which it it easy to find a solution--namely $\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 2 \\ -1 \\ 0 \end{pmatrix}$ (i.e. $x_1=2$, $x_2=-1$, $x_3=0$)

Since we are assured that System #10 is equivalent to our original system, we know that $\begin{pmatrix} 2 \\ -1 \\ 0 \end{pmatrix}$ is the one and only solution to :

$$\begin{aligned} 2x_1 + x_2 + 2x_3 &= 3 \\ x_1 + 3x_2 + 2x_3 &= -1 \\ -x_1 + 2x_2 - x_3 &= -4 \end{aligned}$$

2.3 The Matrix of Coefficients and the Augmented Matrix

Consider again the general system of m simultaneous linear equations in n unknowns:

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \cdots\cdots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + \cdots\cdots + a_{2n}x_n &= b_2 \\ \vdots \\ a_{m1}x_1 + a_{m2}x_2 + a_{m3}x_3 + \cdots\cdots + a_{mn}x_n &= b_m \end{aligned}$$

The m x n matrix

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} \cdots\cdots a_{1n} \\ a_{21} & a_{22} & a_{23} \cdots\cdots a_{2n} \\ \vdots & & \\ a_{m1} & a_{m2} & a_{m3} \cdots\cdots a_{mn} \end{pmatrix}$$

is called the <u>matrix of coefficients</u> for this system.

The m x n+1 (read: m by (n plus one)) matrix

$$\begin{pmatrix} a_{11} & a_{12} & a_{13}\cdots\cdots\cdots a_{1n} & b_1 \\ a_{21} & a_{22} & a_{23}\cdots\cdots\cdots a_{2n} & b_2 \\ \vdots & & & \\ a_{m1} & a_{m2} & a_{m3}\cdots\cdots\cdots a_{mn} & b_m \end{pmatrix}$$

is called the augmented matrix of this system.

Given a system of simultaneous linear equations, it is a simple matter to construct the augmented matrix which represents this system. For example, the system

$$\begin{aligned} 3x_1 + 5x_2 - 7x_3 + x_4 &= 2 \\ 2x_2 + x_3 - 8x_4 &= 20 \\ 6x_1 - x_2 + x_4 &= 0 \end{aligned}$$

is represented by the augmented matrix

$$\left(\begin{array}{cccc|c} 3 & 5 & -7 & 1 & 2 \\ 0 & 2 & 1 & -8 & 20 \\ 6 & -1 & 0 & 1 & 0 \end{array}\right)$$

where we include the bar merely to remind ourselves that the elements to the left of the bar are the coefficients of the unknowns in our system, while the elements to the right of the bar are the constants on the right side of the equal signs in our system.

Likewise, it is a simple matter to construct the system of linear equations which corresponds to a given augmented matrix.

Let us now look at the augmented matrices which represent the equivalent systems of simultaneous linear equations which we derived from our original system #1(Section 2.2)

$$\begin{aligned} 2x_1 + x_2 + 2x_3 &= 3 \\ x_1 + 3x_2 + 2x_3 &= -1 \\ -x_1 + 2x_2 - x_3 &= -4 \end{aligned}$$

The augmented matrix of this system is

$$\left(\begin{array}{ccc|c} 2 & 1 & 2 & 3 \\ 1 & 3 & 2 & -1 \\ -1 & 2 & -1 & -4 \end{array}\right)$$

Likewise the augmented matrices of the equivalent systems 2 through 10 are:

#2 $$\left(\begin{array}{ccc|c} 1 & 3 & 2 & -1 \\ 2 & 1 & 2 & 3 \\ -1 & 2 & -1 & -4 \end{array}\right)$$

#3 $$\left(\begin{array}{ccc|c} 1 & 3 & 2 & -1 \\ 0 & -5 & -2 & 5 \\ -1 & 2 & -1 & -4 \end{array}\right)$$

#4 $$\left(\begin{array}{ccc|c} 1 & 3 & 2 & -1 \\ 0 & -5 & -2 & 5 \\ 0 & 5 & 1 & -5 \end{array}\right)$$

#5 $$\left(\begin{array}{ccc|c} 1 & 3 & 2 & -1 \\ 0 & 1 & 2/5 & -1 \\ 0 & 1 & 5 & -5 \end{array}\right)$$

#6 $$\left(\begin{array}{ccc|c} 1 & 0 & 4/5 & 2 \\ 0 & 1 & 2/5 & -1 \\ 0 & 1 & 5 & -5 \end{array}\right)$$

#7
$$\left(\begin{array}{ccc|c} 1 & 0 & 4/5 & 2 \\ 0 & 1 & 2/5 & -1 \\ 0 & 0 & -1 & 0 \end{array}\right)$$

#8
$$\left(\begin{array}{ccc|c} 1 & 0 & 4/5 & 2 \\ 0 & 1 & 2/5 & -1 \\ 0 & 0 & 1 & 0 \end{array}\right)$$

#9
$$\left(\begin{array}{ccc|c} 1 & 0 & 4/5 & 2 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{array}\right)$$

#10
$$\left(\begin{array}{ccc|c} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{array}\right)$$

The system which corresponds to our final matrix is

$$\begin{array}{l} 1x_1 + 0x_2 + 0x_3 = \ \ 2 \\ 0x_2 + 1x_2 + 0x_3 = -1 \\ 0x_1 + 0x_2 + 1x_3 = \ \ 0 \end{array} \qquad \text{or} \qquad \begin{array}{cccl} x_1 & & & = 2 \\ & x_2 & & = -1 \\ & & x_3 & = 0 \end{array}$$

which we trivially solve to obtain: $\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 2 \\ -1 \\ 0 \end{pmatrix}$

Hence we see that to solve a system of simultaneous linear equations, we merely construct its augmented matrix and then proceed to put this matrix in its row reduced echelon form. We then construct the system which corresponds to our final matrix and "solve" this system. Since the system which we construct from our final matrix will be equivalent to our original system, we will have "solved our original system as well.

Examples:

(1) Solve:

$$
\begin{aligned}
x_1 - x_2 + x_3 &= 7 \\
3x_1 + 2x_2 - x_3 &= -4 \\
2x_1 + x_2 + 2x_3 &= 11
\end{aligned}
$$

We construct the augmented matrix $\left(\begin{array}{ccc|c} 1 & -1 & 1 & 7 \\ 3 & 2 & -1 & -4 \\ 2 & 1 & 2 & 11 \end{array}\right)$

and proceed to put it in its row reduced echelon form.

$$
\left(\begin{array}{ccc|c} 1 & -1 & 1 & 7 \\ 3 & 2 & -1 & -4 \\ 2 & 1 & 2 & 11 \end{array}\right)
\sim
\left(\begin{array}{ccc|c} 1 & -1 & 1 & 7 \\ 0 & 5 & -4 & -25 \\ 0 & 3 & 0 & -3 \end{array}\right)
$$

$$
\sim
\left(\begin{array}{ccc|c} 1 & -1 & 1 & 7 \\ 0 & 1 & -4/5 & -5 \\ 0 & 3 & 0 & -3 \end{array}\right)
$$

$$
\sim
\left(\begin{array}{ccc|c} 1 & 0 & 1/5 & 2 \\ 0 & 1 & -4/5 & -5 \\ 0 & 0 & 12/5 & 12 \end{array}\right)
$$

$$
\sim
\left(\begin{array}{ccc|c} 1 & 0 & 1/5 & 2 \\ 0 & 1 & -4/5 & -5 \\ 0 & 0 & 1 & 5 \end{array}\right)
$$

$$\sim \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 5 \end{pmatrix}$$

We now construct the system represented by our final matrix, assured that it will be equivalent to our original system.

$$\begin{aligned} x_1 \qquad\qquad &= 1 \\ x_2 \qquad &= -1 \\ x_3 &= 5 \end{aligned}$$

We solve this system trivially to obtain $x_1 = 1$, $x_2 = -1$, $x_3 = 5$ or

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \\ 5 \end{pmatrix}$$

(2) Solve:

$$\begin{aligned} x_1 + 2x_2 + 3x_3 + x_4 &= 3 \\ 3x_1 + 2x_2 + x_3 + x_4 &= 7 \\ 2x_2 + 4x_3 + x_4 &= 1 \\ x_1 + x_2 + x_3 + x_4 &= 4 \end{aligned}$$

Constructing the augmented matrix for this system and then putting it in its row reduced echelon form, we have:

$$\left(\begin{array}{cccc|c} 1 & 2 & 3 & 1 & 3 \\ 3 & 2 & 1 & 1 & 7 \\ 0 & 2 & 4 & 1 & 1 \\ 1 & 1 & 1 & 1 & 4 \end{array}\right) \qquad \left(\begin{array}{cccc|c} 1 & 2 & 3 & 1 & 3 \\ 0 & -4 & -8 & -2 & -2 \\ 0 & 2 & 4 & 1 & 1 \\ 0 & -1 & -2 & 0 & 1 \end{array}\right)$$

$$\left(\begin{array}{cccc|c} 1 & 2 & 3 & 1 & 3 \\ 0 & -1 & -2 & 0 & 1 \\ 0 & 2 & 4 & 1 & 1 \\ 0 & -4 & -8 & -2 & -2 \end{array}\right)$$

$$\sim \begin{pmatrix} 1 & 2 & 3 & 1 & 3 \\ 0 & 1 & 2 & 0 & -1 \\ 0 & 2 & 4 & 1 & 1 \\ 0 & -4 & -8 & -2 & -2 \end{pmatrix}$$

$$\sim \begin{pmatrix} 1 & 0 & -1 & 1 & 5 \\ 0 & 1 & 2 & 0 & -1 \\ 0 & 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & -2 & -6 \end{pmatrix}$$

$$\sim \begin{pmatrix} 1 & 0 & -1 & 0 & 2 \\ 0 & 1 & 2 & 0 & -1 \\ 0 & 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

The system which corresponds to our final matrix is

$$\begin{aligned} 1x_1 + 0x_2 - 1x_3 + 0x_4 &= 2 \\ 0x_1 + 1x_2 + 2x_3 + 0x_4 &= -1 \\ 0x_1 + 0x_2 + 0x_3 + 1x_4 &= 3 \\ 0x_1 + 0x_2 + 0x_3 + 0x_4 &= 0 \end{aligned}$$

or

$$\begin{aligned} x_1 \qquad - x_3 \qquad &= 2 \\ x_2 + 2x_3 \qquad &= -1 \\ x_4 &= 3 \end{aligned}$$

If we let x_3 take on <u>any</u> value, say $x_3=t$, we have $x_1=2+t$, $x_2=-1+2t$, $x_3=t$, and $x_4=3$. That is for any value of t,

$$\begin{pmatrix} 2 + t \\ -1 + 2t \\ t \\ 3 \end{pmatrix}$$

is a solution to our system. For example,

for t=o, we obtain the solution $\begin{pmatrix} 2 \\ -1 \\ 0 \\ 3 \end{pmatrix}$

for t=1, we obtain the solution $\begin{pmatrix} 3 \\ 1 \\ 1 \\ 3 \end{pmatrix}$

for t=-7/8, we obtain the solution $\begin{pmatrix} 9/8 \\ -11/4 \\ -7/8 \\ 3 \end{pmatrix}$

(3) Solve:

$$\begin{aligned} 2x_1 + 4x_2 + 2x_3 - 2x_4 + 4x_5 &= 10 \\ 3x_1 + 5x_2 + 7x_3 + 4x_4 - x_5 &= 3 \end{aligned}$$

Constructing the augmented matrix for this example and putting into its row reduced echelon form we have:

$$\left(\begin{array}{ccccc|c} 2 & 4 & 2 & -2 & 4 & 10 \\ 3 & 5 & 7 & 4 & -1 & 3 \end{array}\right) \backsim \left(\begin{array}{ccccc|c} 1 & 2 & 1 & -1 & 2 & 5 \\ 3 & 5 & 7 & 4 & -1 & 3 \end{array}\right)$$

$$\backsim \left(\begin{array}{ccccc|c} 1 & 2 & 1 & -1 & 2 & 5 \\ 0 & -1 & 4 & 7 & -7 & -12 \end{array}\right)$$

$$\backsim \left(\begin{array}{ccccc|c} 1 & 2 & 1 & -1 & 2 & 5 \\ 0 & 1 & -4 & -7 & 7 & 12 \end{array}\right)$$

$$\backsim \left(\begin{array}{ccccc|c} 1 & 0 & 9 & 13 & -12 & -19 \\ 0 & 1 & -4 & -7 & 7 & 12 \end{array}\right)$$

The system which corresponds to our final matrix is:

$$\begin{aligned} x_1 \qquad + 9x_3 + 13x_4 - 12x_5 &= -19 \\ x_2 - 4x_3 - 7x_4 + 7x_5 &= 12 \end{aligned}$$

If we let x_3, x_4, and x_5, take on any values, say $x_3=t$, $x_4=u$, and $x_5=v$, we obtain a solution to our system. That is

$$\begin{pmatrix} -19 - 9t - 13u + 12v \\ 12 + 4t + 7u - 7v \\ t \\ u \\ v \end{pmatrix}$$

is a solution to our system for any values t, u, and v.

In the previous three examples, we have seen that a system of simultaneous linear equations may have one solution or many solutions. Now consider the following system.

(4) Solve:

$$\begin{aligned} x_1 - 2x_2 + x_3 &= 1 \\ 2x_1 - 5x_2 - x_3 &= 5 \\ -x_1 + x_2 - 4x_3 &= 3 \end{aligned}$$

We construct the augmented matrix and proceed to put it in its row reduced echelon form:

$$\left(\begin{array}{ccc|c} 1 & -2 & 1 & 1 \\ 2 & -5 & -1 & 5 \\ -1 & 1 & -4 & 3 \end{array}\right) \sim \left(\begin{array}{ccc|c} 1 & -2 & 1 & 1 \\ 0 & -1 & -3 & 3 \\ 0 & -1 & -3 & 4 \end{array}\right)$$

$$\sim \left(\begin{array}{ccc|c} 1 & -2 & 1 & 1 \\ 0 & 1 & 3 & -3 \\ 0 & -1 & -3 & 4 \end{array}\right)$$

$$\sim \left(\begin{array}{ccc|c} 1 & 0 & 7 & -5 \\ 0 & 1 & 3 & -3 \\ 0 & 0 & 0 & 1 \end{array}\right)$$

$$\left(\begin{array}{ccc|c} 1 & 0 & 7 & 0 \\ 0 & 1 & 3 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right)$$

Our final matrix corresponds to the system:

$$\begin{aligned} 1x_1 + 0x_2 + 7x_3 &= 0 \\ 0x_1 + 1x_2 + 3x_3 &= 0 \\ 0x_1 + 0x_2 + 0x_3 &= 1 \end{aligned}$$

A look at the last equation of this system tells us that this system (and hence our original system) has no solution. That is, there are no real numbers x_1, x_2, and x_3 such that

$$0x_1 + 0x_2 + 0x_3 = 1$$

Let us take a closer look at the matrices involved in this system. For our matrix of coefficients and its row reduced echelon form, we have

$$\begin{pmatrix} 1 & -2 & 1 \\ 2 & -5 & -1 \\ -1 & 1 & -4 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 7 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \end{pmatrix}$$

while for our augmented matrix and its row reduced echelon form, we have

$$\begin{pmatrix} 1 & -2 & 1 & 1 \\ 2 & -5 & -1 & 5 \\ -1 & 1 & -4 & 3 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 7 & 0 \\ 0 & 1 & 3 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Hence, by examining the row reduced echelon forms of the matrix of coefficients and the augmented matrix, we see that:

The rank of the matrix of coefficients = 2

$\neq$ The rank of the augmented matrix = 3.

This is true in general. That is, it can be shown (Cullen, page 55) that:

* * * A system of simultaneous linear equations has at least one solution if and only if the rank of the matrix of coefficients is equal to the rank of the augmented matrix. * * *

2.4 Matrix Representation

Consider the system:

$$\begin{aligned} 3x_1 + 2x_2 + x_3 - x_4 + 7x_5 &= 10 \\ x_1 - x_2 - 4x_3 \quad + x_5 &= 8 \\ -2x_1 + x_2 + 5x_3 + x_4 - x_5 &= 15 \end{aligned}$$

We may express this system of simultaneous linear equations in matrix notation as

$$AX = B$$

where X is the 5 x 1 matrix of "unknowns" $\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix}$; A is the matrix of coefficients $\begin{pmatrix} 3 & 2 & 1 & -1 & 7 \\ 1 & -1 & -4 & 0 & 1 \\ -2 & 1 & 5 & 1 & -1 \end{pmatrix}$; and B is the 3 x 1 matrix of right-hand side constants $\begin{pmatrix} 10 \\ 8 \\ 15 \end{pmatrix}$.

That is, with the above description of A, X, and B, the matrix equation AX=B becomes:

$$\begin{pmatrix} 3 & 2 & 1 & -1 & 7 \\ 1 & -1 & -4 & 0 & 1 \\ -2 & 1 & 5 & 1 & -1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix} = \begin{pmatrix} 10 \\ 8 \\ 15 \end{pmatrix}$$

Looking at the left side of the above equation, note that we have a 3 x 5 matrix times a 5 x 1 matrix (with the 3 x 5 matrix as the left factor and the 5 x 1 matrix as the right factor). A 3 x 5 matrix times a 5 x 1 matrix (in that order) yields a 3 x 1 matrix which is indeed what we have on the right side of the equal sign. Hence, the above matrix equation makes sense.

Multiplying the 3 x 5 matrix A times the 5 x 1 matrix X to obtain the 3 x 1 matrix AX, we have:

$$AX = \begin{pmatrix} (3x_1 + 2x_2 + x_3 - x_4 + 7x_5) \\ (x_1 - x_2 - 4x_3 + x_5) \\ (-2x_1 + x_2 + 5x_3 + x_4 - x_5) \end{pmatrix}$$

which is to be equal to the 3 x 1 matrix $\begin{pmatrix} 10 \\ 8 \\ 15 \end{pmatrix}$

Recall that for two matrices to be equal, their corresponding elements must be equal. Hence, we must have:

$$\begin{aligned} 3x_1 + 2x_2 + x_3 - x_4 + 7x_5 &= 10 \\ x_1 - x_2 - 4x_3 + x_5 &= 8 \\ -2x_1 + x_2 + 5x_3 + x_4 - x_5 &= 15 \end{aligned}$$

which is our original system.

Consider again the general system of m simultaneous linear equations in n unknown:

$$
\begin{aligned}
&a_{11}x_1 + a_{12}x_2 + \cdots\cdots + a_{1n}x_n = b_1\\
&a_{21}x_1 + a_{22}x_2 + \cdots\cdots + a_{2n}x_n = b_2\\
&\vdots\\
&a_{m1}x_1 + a_{m2}x_2 + \cdots\cdots + a_{mn}x_n = b_m
\end{aligned}
$$

This system written in matrix notation is:

$$
\begin{pmatrix}
a_{11} & a_{12} & a_{13}\cdots\cdots a_{1n}\\
a_{21} & a_{22} & a_{23}\cdots\cdots a_{2n}\\
\vdots & & \\
a_{m1} & a_{m2} & a_{m3}\cdots\cdots a_{mn}
\end{pmatrix}
\begin{pmatrix} x_1\\ x_2\\ \vdots\\ x_n \end{pmatrix}
=
\begin{pmatrix} b_1\\ b_2\\ \vdots\\ b_m \end{pmatrix}
$$

or AX = B, where A is the m x n, matrix of coefficients. X is the n x 1 matrix of unknown and B is the m x 1 matrix of right-hand side constants.

In the terminology to be introduced in the next chapter, we shall call $X = \begin{pmatrix} x_1\\ x_2\\ \vdots\\ x_n \end{pmatrix}$ the <u>solution vector</u> to the above system.

Now <u>suppose that our system of linear equation has the same number of equation as unknowns</u>. That is, our system has the form:

$$
\begin{aligned}
&a_{11}x_1 + a_{12}x_2 + \cdots\cdots + a_{1n}x_n = b_1\\
&a_{21}x_1 + a_{22}x_2 + \cdots\cdots + a_{2n}x_2 = b_2\\
&\vdots\\
&a_{n1}x_1 + a_{n2}x_2 + \cdots\cdots + a_{nn}x_n = b_n
\end{aligned}
$$

Expressing this in matrix notation, we have:

$$AX = B$$

where A is the n x n matrix of coefficients; X is the n x 1 solution matrix (or vector); and B is the n x 1 matrix (or vector) whose elements are the right-hand side constants of the system.

The significant point here is that <u>A is square</u> and hence we may talk about A^{-1} and $|A|$. If $|A| \neq 0$, by 1.5B, A^{-1} exists.

Since	$AX = B$	
we have	$A^{-1}(AX) = A^{-1}B$	by 1.4C
which implies	$(A^{-1}A)X = A^{-1}B$	by 1.4B
from which we have	$(I)X = A^{-1}B$	
which in turn implies that	$X = A^{-1}B$	

Recall that A being n x n implies that A^{-1} is n x n . Since A^{-1} is n x n and B is n x 1, $A^{-1}B$ is n x 1. This is what we expected since X is n x 1 .

A: Hence we see that if our system of simultaneous linear equation in n unknowns is such that A, the matrix of coefficients, is square, and if we further have that $|A| \neq 0$ (which will imply that A^{-1} exists), then we can be sure that our system will have a unique solution

$$X = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}$$

which will be given by

$$X = A^{-1}B$$

where B is as defined before.

Example:

Consider the system of <u>three</u> equations in <u>three</u> unknowns:

$$\begin{aligned} 2x_1 \qquad\quad + x_3 &= 4 \\ -x_1 + 3x_2 + 4x_3 &= 6 \\ 5x_2 \qquad\quad &= -10 \end{aligned}$$

Writing this system in matrix notation, we have AX = B.

A, the matrix of coefficients for this system is

$$\begin{pmatrix} 2 & 0 & 1 \\ -1 & 3 & 4 \\ 0 & 5 & 0 \end{pmatrix}$$

while $B = \begin{pmatrix} 4 \\ 6 \\ -10 \end{pmatrix}$

In section 1.5, we found that $|A| \neq 0$. Hence we have a unique solution to this system given by $X = A^{-1} B$. Again in 1.5, we found that

$$A^{-1} = \begin{pmatrix} 4/9 & -1/9 & 1/15 \\ 0 & 0 & 1/5 \\ 1/9 & 2/9 & -2/15 \end{pmatrix}$$

$$\text{Hence, } X = \begin{pmatrix} 4/9 & -1/9 & 1/15 \\ 0 & 0 & 1/5 \\ 1/9 & 2/9 & -2/15 \end{pmatrix} \begin{pmatrix} 4 \\ 6 \\ -10 \end{pmatrix}$$

$$= \begin{pmatrix} 16/9 & -\ 6/9 & -10/15 \\ & & -10/5 \\ 4/9 & +12/9 & +20/15 \end{pmatrix}$$

$$= \begin{pmatrix} 4/9 \\ -2 \\ 28/9 \end{pmatrix}$$

That is, $x_1 = 4/9$; $x_2 = -2$; $x_3 = 28/9$

Our previous method for solving this system would be to construct the augmented matrix and then put it in its row reduced echelon form. We would have :

$$\left(\begin{array}{ccc|c} 2 & 0 & 1 & 4 \\ -1 & 3 & 4 & 6 \\ 0 & 5 & 0 & -10 \end{array}\right)$$

$$\left(\begin{array}{ccc|c} -1 & 3 & 4 & 6 \\ 2 & 0 & 1 & 4 \\ 0 & 5 & 0 & -10 \end{array}\right)$$

$$\left(\begin{array}{ccc|c} 1 & -3 & -4 & -6 \\ 2 & 0 & 1 & 4 \\ 0 & 5 & 0 & -10 \end{array}\right)$$

$$\left(\begin{array}{ccc|c} 1 & -3 & -4 & -6 \\ 0 & 6 & 9 & 16 \\ 0 & 5 & 0 & -10 \end{array}\right)$$

$$\left(\begin{array}{ccc|c} 1 & -3 & -4 & -6 \\ 0 & 1 & 3/2 & 8/3 \\ 0 & 5 & 0 & -10 \end{array}\right)$$

$$\left(\begin{array}{ccc|c} 1 & 0 & 1/2 & 2 \\ 0 & 1 & 3/2 & 8/3 \\ 0 & 0 & -15/2 & -70/3 \end{array}\right)$$

$$\left(\begin{array}{ccc|c} 1 & 0 & 1/2 & 2 \\ 0 & 1 & 3/2 & 8/3 \\ 0 & 0 & 1 & 28/9 \end{array}\right)$$

$$\left(\begin{array}{ccc|c} 1 & 0 & 0 & 4/9 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & 28/9 \end{array}\right)$$

Hence a system equivalent to our original one is:

$$\begin{aligned} x_1 \qquad\qquad &= 4/9 \\ x_2 \qquad &= -2 \\ x_3 &= 28/9 \end{aligned}$$

which we trivially solve to obtain the solution matrix (or vector)

$$X = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 4/9 \\ -2 \\ 28/9 \end{pmatrix}$$

CHAPTER 3: VECTOR SPACES

3.1 Definitions

Although the mathematical concept of a vector space is very abstract, we shall have need of only a rather narrow interpretation of this concept.

Indeed, rigorously speaking, our definition of "vector space" will be merely an example of this abstract concept.

For the purposes of this manual, the following definition will suffice.

By an n-dimensional <u>vector space</u> we shall mean the set of all n x 1 matrices $V = \begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{pmatrix}$ where, as before, $v_1, v_2, \ldots\ldots, v_n$ are any real numbers. We shall refer to such a column matrix as an n-dimensional vector and shall call v_i the i-th component of V.

We define multiplication of a vector, V, by a real number, c, to yield a vector cV as follows:

$$cV = c\begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{pmatrix} = \begin{pmatrix} cv_1 \\ cv_2 \\ \vdots \\ cv_n \end{pmatrix}$$

The definition of X + Y for vectors X and Y follows from the definition of matrix addition. That is, if $X = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}$ and $Y = \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{pmatrix}$

then the vector $X + Y = \begin{pmatrix} x_1 + y_1 \\ x_2 + y_2 \\ \vdots \\ x_n + y_n \end{pmatrix}$.

We define $X \cdot Y$, the "dot product" of two n-dimensional vectors X and Y as follows

$$X \cdot Y = x_1y_1 + x_2y_2 + x_3y_3 + \dots\dots + x_ny_n$$

That is, the dot product of two vectors is a real number. Note that this is not the same as matrix multiplication.

Indeed the <u>matrix product</u>, X x Y, of the two vectors (i.e., 1 x n matrices) is not even defined since the number of columns in X is equal to n while the number of rows in Y is equal to one.

Examples:

Let $X = \begin{pmatrix} 2 \\ 0 \\ -1 \\ 7 \end{pmatrix}$; $Y = \begin{pmatrix} 8 \\ 5 \\ 9 \\ 4 \end{pmatrix}$; $c = 3$ then

$X + Y = \begin{pmatrix} 10 \\ 5 \\ 8 \\ 11 \end{pmatrix}$; $cX = \begin{pmatrix} 6 \\ 0 \\ -3 \\ 21 \end{pmatrix}$

We shall say that the k-n-dimensional vectors $V_1, V_2, V_3, \dots, V_k$ are <u>linearly dependent</u> if there exists k real numbers $c_1, c_2, c_3, \dots, c_k$, such that:

$$c_1V_1 + c_2V_2 + c_3V_3 + \dots\dots + c_kV_k = 0$$

where by the "0" on the right-hand side we mean the vector whose n components are all zero.

The expression on the left side of the equal sign is called a linear combination of the vectors $V_1, V_2, \ldots\ldots, V_k$.

Note that the above vector equation makes sense since on the right, 0 is an n- dimensional vector; and on the left, the V_i being n-dimensional vectors implies that each c_iV_i is an n-dimensional vector which in turn implies that their sum is an n-dimensional vector. Hence we have an n-dimensional vector represented on each side of the equal sign.

We say that the K vectors $V_1, V_2, \ldots\ldots, V_k$ are linearly independent if they are not linearly dependent. This implies the following equivalent definition.

The k vectors $V_1, V_2, \ldots\ldots, V_k$ are linearly independent if an equation of the form

$$c_1V_1 + c_2V_2 + \ldots\ldots\ldots + c_kV_k = 0 \qquad \text{(0: an n=dimensional vector)}$$

implies that $c_1 = c_2 = \ldots\ldots\ldots = c_k = 0$ (0: a real number)

Examples:

(1) $\begin{pmatrix} 1 \\ -1 \end{pmatrix}$, $\begin{pmatrix} 4 \\ 2 \end{pmatrix}$, and $\begin{pmatrix} 10 \\ 14 \end{pmatrix}$

are linearly dependent since there exists real numbers, 6, -4, and 1 such that

$$6\begin{pmatrix} 1 \\ -1 \end{pmatrix} - 4\begin{pmatrix} 4 \\ 2 \end{pmatrix} + 1\begin{pmatrix} 10 \\ 14 \end{pmatrix}$$

$$= \begin{pmatrix} 6 \\ -6 \end{pmatrix} + \begin{pmatrix} -16 \\ -8 \end{pmatrix} + \begin{pmatrix} 10 \\ 14 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} = 0$$

The reader may wonder how the numbers 6, -4, and 1 were determined. Keep in mind that we were looking for real numbers c_1, c_2, c_3 such that

$$c_1 \begin{pmatrix} 1 \\ -1 \end{pmatrix} + c_2 \begin{pmatrix} 4 \\ 2 \end{pmatrix} + c_3 \begin{pmatrix} 10 \\ 14 \end{pmatrix} = 0$$

where the 0 on the right-hand side represents the two-dimensional zero vector $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$.

Multiplying by the indicated real numbers and adding the resulting vectors, we have:

$$\begin{pmatrix} (c_1 + 4c_2 + 10c_3) \\ (-c_1 + 2c_2 + 14c_3) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

The definition of equality of vectors follows from the definition of equality of matrices. That is, corresponding elements must be equal. We must therefore have:

$$\begin{aligned} c_1 + 4c_2 + 10c_3 &= 0 \\ -c_1 + 2c_2 + 14c_3 &= 0 \end{aligned}$$

We proceed to solve this system of simultaneous linear equations by our usual method of constructing its augmented matrix and putting it in its row reduced echelon form. We have:

$$\left(\begin{array}{ccc|c} 1 & 4 & 10 & 0 \\ -1 & 2 & 14 & 0 \end{array}\right) \sim \left(\begin{array}{ccc|c} 1 & 4 & 10 & 0 \\ 0 & 6 & 24 & 0 \end{array}\right)$$

$$\sim \left(\begin{array}{ccc|c} 1 & 4 & 10 & 0 \\ 0 & 1 & 4 & 0 \end{array}\right)$$

$$\sim \left(\begin{array}{ccc|c} 1 & 0 & -6 & 0 \\ 0 & 1 & 4 & 0 \end{array}\right)$$

Hence, a system equivalent to our original system can be read from our final matrix:

$$\begin{aligned} c_1 \qquad - 6c_3 &= 0 \\ c_2 + 4c_3 &= 0 \end{aligned}$$

we can therefore let c_3 take on any value, say $c_3 = t$, and solve for c_1 and c_2 in terms of t. That is $c_1 = 6t$, $c_2 = -4t$, $c_3 = t$ will give us a solution to our system for any value of t. Thus, letting $t = 1$ we have $c_1 = 6$, $c_2 = -4$, and $c_3 = 1$.

(2) $\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$, $\begin{pmatrix} 0 \\ -1 \\ 2 \end{pmatrix}$, and $\begin{pmatrix} 3 \\ 1 \\ 0 \end{pmatrix}$

are linearly independent; for suppose we had real numbers c_1, c_2, and c_3 such that

$$c_1 \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} + c_2 \begin{pmatrix} 0 \\ -1 \\ 2 \end{pmatrix} + c_3 \begin{pmatrix} 3 \\ 1 \\ 0 \end{pmatrix} = 0$$

Example:

$$\begin{pmatrix} (c_1 + 3c_3) \\ (-c_2 + c_3) \\ (c_1 + 2c_2) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

This implies that

$$\begin{aligned} c_1 \qquad\quad + 3c_3 &= 0 \\ - c_2 + \; c_3 &= 0 \\ c_1 + 2c_2 \qquad\quad &= 0 \end{aligned}$$

Solving this system by our usual procedure of constructing its augmented matrix, we have:

$$\left(\begin{array}{ccc|c} 1 & 0 & 3 & 0 \\ 0 & -1 & 1 & 0 \\ 1 & 2 & 0 & 0 \end{array}\right) \sim \left(\begin{array}{ccc|c} 1 & 0 & 3 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 2 & -3 & 0 \end{array}\right)$$

$$\sim \left(\begin{array}{ccc|c} 1 & 0 & 3 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 2 & -3 & 0 \end{array}\right)$$

$$\sim \left(\begin{array}{ccc|c} 1 & 0 & 3 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & -1 & 0 \end{array}\right)$$

$$\sim \left(\begin{array}{ccc|c} 1 & 0 & 3 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right)$$

$$\sim \left(\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right)$$

Hence, a system equivalent to our original one can be read from the final matrix:

$$\begin{aligned} c_1 \qquad\qquad &= 0 \\ c_2 \qquad &= 0 \\ c_3 &= 0 \end{aligned}$$

which can be trivially solved to obtain the unique solution:

$$c_1 = c_2 = c_3 = o$$

Thus, we have shown that if

$$c_1 \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} + c_2 \begin{pmatrix} 0 \\ -1 \\ 2 \end{pmatrix} + c_3 \begin{pmatrix} 3 \\ 1 \\ 0 \end{pmatrix} = 0$$

then $c_1 = c_2 = c_3 = 0$. Hence $\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$; $\begin{pmatrix} 0 \\ -1 \\ 2 \end{pmatrix}$; and $\begin{pmatrix} 3 \\ 1 \\ 0 \end{pmatrix}$ are linearly independent.

Note that if we form a matrix using the above 3 vectors as our three column, we obtain a matrix:

$$A = \begin{pmatrix} 1 & 0 & 3 \\ 0 & -1 & 1 \\ 1 & 2 & 0 \end{pmatrix}$$

for which
$$\begin{aligned} |A| &= (-1)^{1+1} (1) \begin{vmatrix} -1 & 1 \\ 2 & 0 \end{vmatrix} + (-1)^{1+2} (0) \begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix} + (-1)^{1+3} (3) \begin{vmatrix} 0 & -1 \\ 1 & 2 \end{vmatrix} \\ &= (0 - 2) + 0 + 3(0 - (-1)) \\ &= 1 \neq 0 \end{aligned}$$

This will be true in general. That is, it can be shown that (Cullen, page 55) :

A: The k k-dimensional $V_1, V_2, \ldots\ldots, V_k$ are linearly independent if the k x k matrix, A, whose columns are these given k vectors, has a determinant which is not equal to zero. (i.e. $|A| \neq 0$)

Conversely:

B: If the k k-dimensional vectors $V_1, V_2, \ldots\ldots, V_k$ are linearly independent, then the k x k matrix described above has a nonzero determinant.

It follows from the above that if the k vectors $V_1, V_2, V_3, \ldots\ldots, V_k$ are linearly <u>dependent</u>, then $|A| = 0$ where $|A|$ is as described above.

Likewise, if the k x k matrix A is such that $|A| = 0$, then the columns of A (considered as vectors) are linearly dependent.

3.2 Basis for a Vector Space

By a basis for an n-dimensional vector space we mean a set of n linearly independent vectors $V_1, V_2, \ldots, V_n$ such that for any n-dimensional vector, V, there exists real numbers $c_1, c_2, \ldots, c_n$ such that

$$V = c_1V_1 + c_2V_2 + \ldots + c_nV_n$$

That is, any vector in the vector space can be expressed as a linear combination of the basis vectors.

Consider the following n vectors in our n-dimensional vector space:

$$e_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}; \quad e_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}; \quad e_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}; \ldots\ldots; \quad e_{n-1} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 1 \\ 0 \end{pmatrix}; \quad e_n = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix}$$

These n vectors are obviously linearly independent, since if there are $c_1\ c_2, \ldots, c_n$ such that

$$c_1e_1 + c_2e_2 + \ldots + c_ne_n = 0 = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix},$$

then we would have

$$\begin{pmatrix} 1c_1 + 0c_2 + 0c_3 + \cdots\cdots + 0c_n \\ 0c_1 + 1c_2 + 0c_3 + \cdots\cdots + 0c_n \\ 0c_1 + 0c_2 + 1c_3 + \cdots\cdots + 0c_n \\ \vdots \\ 0c_1 + 0c_2 + 0c_3 + \cdots\cdots + 1c_n \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

or

$$\begin{aligned} c_1 \quad\quad\quad\quad &= 0 \\ c_2 \quad\quad\quad &= 0 \\ c_3 \quad\quad &= 0 \\ c_n &= 0 \end{aligned}$$

which implies that $c_1 = c_2 = \cdots\cdots = c_n = 0$.

Hence, the vectors $e_1, e_2, e_3, \ldots\ldots, e_n$ are linearly independent.

Note that by a previous theorem (3.1A), we could have shown that $e_1, e_2, \ldots\ldots\ldots, e_n$ are linearly independent by considering the determinant of A, where the columns of A are the n vectors $e_1, e_2, \ldots\ldots\ldots, e_n$. The linear independence of $e_1, e_2, \ldots\ldots\ldots, e_n$ would have followed from the fact that $|A| \neq 0$.

Now let $V = \begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{pmatrix}$ be an arbitrary n-dimensional vector. It is obvious that V can be expressed as the linear combination:

$$v_1e_1 + v_2e_2 + v_3e_3 + \cdots\cdots + v_ne_n$$

Example:

The five dimensional vector $\begin{pmatrix} 2 \\ 17 \\ -3 \\ 4 \\ 8 \end{pmatrix}$ can be written as:

$$2\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} + 17\begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} - 3\begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} + 4\begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} + 8\begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

A: A very useful result (Cullen, page 52) is that <u>any</u> n linearly independent n-dimensional vectors $V_1, V_2, \ldots\ldots V_n$ can serve as a basis for an n-dimensional vector space.

Example:

Consider the three vectors $\begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}$, $\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$, and $\begin{pmatrix} 0 \\ 1 \\ -2 \end{pmatrix}$

by 3.1A, since $\begin{vmatrix} 1 & 1 & 0 \\ -1 & 2 & 1 \\ 0 & 3 & -2 \end{vmatrix} = 1(-1)^{1+1}\begin{vmatrix} 2 & 1 \\ 3 & -2 \end{vmatrix} + 1(-1)^{1+2}\begin{vmatrix} -1 & 1 \\ 0 & -2 \end{vmatrix}$

$$= (-4 - 3) - (2 - 0)$$

$$= -9 \neq 0$$

these three vectors are linearly independent.

Hence, by 3.2A, $\begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}$, $\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$, and $\begin{pmatrix} 0 \\ 1 \\ -2 \end{pmatrix}$

form a basis for a three dimensional vector space.

Consider the vector $\begin{pmatrix} 2 \\ 3 \\ -1 \end{pmatrix}$. From the definition of basis for a vector space, we should be able to express the vector $\begin{pmatrix} 2 \\ 3 \\ -1 \end{pmatrix}$ as a linear combination of the basis vectors $\begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}$, $\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$, and $\begin{pmatrix} 0 \\ 1 \\ -2 \end{pmatrix}$.

That is, there exist real numbers c_1, c_2, c_3 such that

$$\begin{pmatrix} 2 \\ 3 \\ -1 \end{pmatrix} = c_1 \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} + c_2 \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} + c_3 \begin{pmatrix} 0 \\ 1 \\ -2 \end{pmatrix}$$

$$\text{or} \quad \begin{pmatrix} 2 \\ 3 \\ -1 \end{pmatrix} = \begin{pmatrix} (c_1 + c_2) \\ (-c_1 + 2c_2 + c_3) \\ (3c_2 - 2c_3) \end{pmatrix}$$

By the definition of equality of vectors, we have the following system of simultaneous linear equations:

$$\begin{aligned} c_1 + c_2 &= 2 \\ -c_1 + 2c_2 + c_3 &= 3 \\ 3c_2 - 2c_3 &= -1 \end{aligned}$$

We proceed to solve this system by our usual method of constructing the augmented matrix and then putting it in its row reduced echelon form:

$$\left(\begin{array}{ccc|c} 1 & 1 & 0 & 2 \\ -1 & 2 & 1 & 3 \\ 0 & 3 & -2 & -1 \end{array}\right) \sim \left(\begin{array}{ccc|c} 1 & 1 & 0 & 2 \\ 0 & 3 & 1 & 5 \\ 0 & 3 & -2 & -1 \end{array}\right)$$

$$\sim \left(\begin{array}{ccc|c} 1 & 1 & 0 & 2 \\ 0 & 1 & 1/3 & 5/3 \\ 0 & 3 & -2 & -1 \end{array}\right)$$

$$\left(\begin{array}{ccc|c} 1 & 0 & -1/3 & 1/3 \\ 0 & 1 & 1/3 & 5/3 \\ 0 & 0 & -3 & -6 \end{array}\right)$$

$$\left(\begin{array}{ccc|c} 1 & 0 & -1/3 & 1/3 \\ 0 & 1 & 1/3 & 5/3 \\ 0 & 0 & 1 & 2 \end{array}\right)$$

$$\left(\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 2 \end{array}\right)$$

We read the system which is equivalent to our original one from the last matrix above to obtain:

$$\begin{aligned} c_1 \qquad\qquad &= 1 \\ c_2 \qquad &= 1 \\ c_3 &= 2 \end{aligned}$$

for which we trivially solve to obtain:

$$c_1 = 1; \quad c_2 = 1; \quad c_3 = 2$$

Hence $\begin{pmatrix} 2 \\ 3 \\ -1 \end{pmatrix}$ may be written as the following linear combination of the basis vectors $\begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}$; $\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$; $\begin{pmatrix} 0 \\ 1 \\ -2 \end{pmatrix}$:

$$\begin{pmatrix} 2 \\ 3 \\ 1 \end{pmatrix} = 1 \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} + 1 \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} + 2 \begin{pmatrix} 0 \\ 1 \\ -2 \end{pmatrix} .$$

Suppose now that we wanted to express each of the following four vectors:

$$W = \begin{pmatrix} 2 \\ 3 \\ -1 \end{pmatrix} ; \quad X = \begin{pmatrix} 4 \\ 2 \\ 0 \end{pmatrix} ; \quad Y = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} ; \quad Z = \begin{pmatrix} -3 \\ 5 \\ 1 \end{pmatrix}$$

in terms of the above basis vectors:

$$V_1 = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}, \quad V_2 = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}, \quad \text{and} \quad V_3 = \begin{pmatrix} 0 \\ 1 \\ -2 \end{pmatrix}$$

There are, of course, four individual problems here. We must find real numbers a_1, a_2, and a_3 such that

$$\begin{pmatrix} 2 \\ 3 \\ -1 \end{pmatrix} = a_1 \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} + a_2 \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} + a_3 \begin{pmatrix} 0 \\ 1 \\ -2 \end{pmatrix}$$

and real numbers b_1, b_2, and b_3 such that

$$\begin{pmatrix} 4 \\ 2 \\ 0 \end{pmatrix} = b_1 \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} + b_2 \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} + b_3 \begin{pmatrix} 0 \\ 1 \\ -2 \end{pmatrix}$$

and real numbers c_1, c_2, and c_3 such that

$$\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = c_1 \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} + c_2 \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} + c_3 \begin{pmatrix} 0 \\ 1 \\ -2 \end{pmatrix}$$

and real numbers d_1, d_2, and d_3 such that

$$\begin{pmatrix} -3 \\ 5 \\ 1 \end{pmatrix} = d_1 \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} + d_2 \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} + d_3 \begin{pmatrix} 0 \\ 1 \\ -2 \end{pmatrix}$$

This gives rise to the following four systems of simultaneous linear equations:

$$(1)\quad \begin{aligned} a_1 + a_2 &= 2 \\ -a_1 + 2a_2 + a_3 &= 3 \\ 3a_2 - 2a_3 &= -1 \end{aligned}$$

$$(2)\quad \begin{aligned} b_1 + b_2 &= 4 \\ -b_1 + 2b_2 + b_3 &= 2 \\ 3b_2 - 2b_3 &= 0 \end{aligned}$$

$$(3)\quad \begin{aligned} c_1 + c_2 &= 0 \\ -c_1 + 2c_2 + c_3 &= 1 \\ 3c_2 - 2c_3 &= 0 \end{aligned}$$

$$(4)\quad \begin{aligned} d_1 + d_2 &= -3 \\ -d_1 + 2d_2 + d_3 &= 5 \\ 3d_2 - 2d_3 &= 1 \end{aligned}$$

which in turn gives rise to the following four augmented matrices:

$$(1)\quad \left(\begin{array}{ccc|c} 1 & 1 & 0 & 2 \\ -1 & 2 & 1 & 3 \\ 0 & 3 & -2 & -1 \end{array}\right)$$

$$(2)\quad \left(\begin{array}{ccc|c} 1 & 1 & 0 & 4 \\ -1 & 2 & 1 & 2 \\ 0 & 3 & -2 & 0 \end{array}\right)$$

$$(3)\quad \left(\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ -1 & 2 & 1 & 1 \\ 0 & 3 & -2 & 0 \end{array}\right)$$

$$(4)\quad \left(\begin{array}{ccc|c} 1 & 1 & 0 & -3 \\ -1 & 2 & 1 & 5 \\ 0 & 3 & -2 & 1 \end{array}\right)$$

Finding the row reduced echelon forms of the above augmented matrices, we have:

$$(1) \quad \left(\begin{array}{ccc|c} 1 & 1 & 0 & 2 \\ -1 & 2 & 1 & 3 \\ 0 & 3 & -2 & -1 \end{array}\right) \sim \left(\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 2 \end{array}\right)$$

$$(2) \quad \left(\begin{array}{ccc|c} 1 & 1 & 0 & 4 \\ -1 & 2 & 1 & 2 \\ 0 & 3 & -2 & 0 \end{array}\right) \sim \left(\begin{array}{ccc|c} 1 & 0 & 0 & 8/3 \\ 0 & 1 & 0 & 4/3 \\ 0 & 0 & 1 & 2 \end{array}\right)$$

$$(3) \quad \left(\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ -1 & 2 & 1 & 1 \\ 0 & 3 & -2 & 0 \end{array}\right) \quad \left(\begin{array}{ccc|c} 1 & 0 & 0 & -2/9 \\ 0 & 1 & 0 & 2/9 \\ 0 & 0 & 1 & 1/3 \end{array}\right)$$

$$(4) \quad \left(\begin{array}{ccc|c} 1 & 1 & 0 & -3 \\ -1 & 2 & 1 & 5 \\ 0 & 3 & -2 & 1 \end{array}\right) \sim \left(\begin{array}{ccc|c} 1 & 0 & 0 & -32/9 \\ 0 & 1 & 0 & 5/9 \\ 0 & 0 & 1 & 1/3 \end{array}\right)$$

We would then construct the four systems corresponding to these matrices. These systems would be equivalent to the original four systems.

We have seen, however, that solving the systems represented by our four augmented matrices in row reduced echelon form amounts to merely reading the elements in the last column of these matrices.

Hence, $a_1 = 1$, $a_2 = 1$, $a_3 = 2$, b_1 8/3, $b_2 = 4/3$, $b_3 = 2$, $c_1 = -2/9$, $c_2 = 2/9$, $c_3 = 1/3$, $d_1 = -32/9$, $d_2 = 5/9$, $d_3 = 1/3$.

Leaving out the step of constructing and trivially solving the systems represented by the augmented matrices in row reduced echelon form, we can shorten our work by symbolically combining the above four problems.

It will often occur in our work in Part II that we wish to express each of a given set of vectors in terms of some set of a few basis vectors.

Let us return to our original problem. Express each of the vectors

$$W = \begin{pmatrix} 2 \\ 3 \\ -1 \end{pmatrix}, \quad X = \begin{pmatrix} 4 \\ 2 \\ 0 \end{pmatrix}, \quad Y = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \quad Z = \begin{pmatrix} -3 \\ 5 \\ 1 \end{pmatrix}$$

in terms of the basis vectors: $\begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ -2 \end{pmatrix}$

Examining what we have done before, we arrived at four systems of simultaneous linear equations from which we constructed four 3 x 4 augmented matrices which we proceeded to put in row reduced echelon form.

Let us now simplify our work by constructing one 3 x 7 matrix as follows:

$$\begin{array}{c} \begin{array}{ccc|cccc} V_1 & V_2 & V_3 & W & X & Y & Z \end{array} \\ \left(\begin{array}{ccc|cccc} 1 & 1 & 0 & 2 & 4 & 0 & -3 \\ -1 & 2 & 1 & 3 & 2 & 1 & 5 \\ 0 & 3 & -2 & -1 & 0 & 0 & 1 \end{array}\right) \end{array}$$

where we have designated the vectors represented by each of the columns.

We now put this matrix in row reduced echelon form.

$$\begin{array}{c} \begin{array}{ccc|cccc} V_1 & V_2 & V_3 & W & X & Y & Z \end{array} \\ \left(\begin{array}{ccc|cccc} 1 & 1 & 0 & 2 & 4 & 0 & -3 \\ -1 & 2 & 1 & 3 & 2 & 1 & 5 \\ 0 & 3 & -2 & -1 & 0 & 0 & 1 \end{array}\right) \end{array} \sim \begin{array}{c} \begin{array}{ccc|cccc} V_1 & V_2 & V_3 & W & X & Y & Z \end{array} \\ \left(\begin{array}{ccc|cccc} 1 & 1 & 0 & 2 & 4 & 0 & -3 \\ 0 & 3 & 1 & 5 & 6 & 1 & 2 \\ 0 & 3 & -2 & -1 & 0 & 0 & 1 \end{array}\right) \end{array}$$

$$\sim \begin{array}{c} \begin{array}{ccc|cccc} V_1 & V_2 & V_3 & W & X & Y & Z \end{array} \\ \left(\begin{array}{ccc|cccc} 1 & 1 & 0 & 2 & 4 & 0 & -3 \\ 0 & 1 & 1/3 & 5/3 & 2 & 1/3 & 2/3 \\ 0 & 3 & -2 & -1 & 0 & 0 & 1 \end{array}\right) \end{array}$$

$$\begin{array}{c} \begin{array}{ccccccc} V_1 & V_2 & V_3 & W & X & Y & Z \end{array} \\ \left(\begin{array}{ccc|cccc} 1 & 0 & -1/3 & 1/3 & 2 & -1/3 & -11/3 \\ 0 & 1 & 1/3 & 5/3 & 2 & 1/3 & 2/3 \\ 0 & 0 & -3 & -6 & -6 & -1 & -1 \end{array}\right) \end{array}$$

$$\sim \begin{array}{c} \begin{array}{ccccccc} V_1 & V_2 & V_3 & W & X & Y & Z \end{array} \\ \left(\begin{array}{ccc|cccc} 1 & 0 & -1/3 & 1/3 & 2 & -1/3 & -11/3 \\ 0 & 1 & 1/3 & 5/3 & 2 & 1/3 & 2/3 \\ 0 & 0 & 1 & 2 & 2 & 1/3 & 1/3 \end{array}\right) \end{array}$$

$$\sim \begin{array}{c} \begin{array}{ccccccc} V_1 & V_2 & V_3 & W & X & Y & Z \end{array} \\ \left(\begin{array}{ccc|cccc} 1 & 0 & 0 & 1 & 8/3 & -2/9 & -32/9 \\ 0 & 1 & 0 & 1 & 4/3 & 2/9 & 5/9 \\ 0 & 0 & 1 & 2 & 2 & 1/3 & 1/3 \end{array}\right) \end{array}$$

Compare this last matrix with the row reduced echelon forms of our four 3 x 4 augmented matrices. We may therefore determine the coeffieients a_1, a_2, a_3, b_1, b_2, b_3, c_1, c_2, c_3, d_1, d_2, and d_3 from our final 3 x 7 matrix.

We have:

$$W = 1\ V_1 + 1\ V_2 + 2\ V_3$$

$$X = 8/3\ V_1 + 4/3\ V_2 + 2\ V_3$$

$$Y = -2/9\ V_1 + 2/9\ V_2 + 1/3\ V_3$$

$$Z = -32/9\ V_1 + 5/9\ V_2 + 1/3\ V_3$$

It is important that one understand the above procedure since it is crucial for what is to follow in Part II.

Note that if V_1, V_2, V_3, . . . , V_n constitute a basis for an n-dimensional vector space, any one of the vectors, V_i, in this set can be trivially represented as a linear combination of the basis vectors as follows:

$$V_i = 0\ V_1 + 0\ V_2 + \cdots + {}_{i-1} + V_i + 0\ V_{i+1} + \cdots + 0\ V_n$$

Example:

Let $V_1 = \begin{pmatrix} 1 \\ -8 \\ 1 \\ -10 \end{pmatrix}$; $V_2 = \begin{pmatrix} 2 \\ 4 \\ 9 \\ 0 \end{pmatrix}$; $V_3 = \begin{pmatrix} 3 \\ 1 \\ 2 \\ 3 \end{pmatrix}$; $V_4 = \begin{pmatrix} 7 \\ 0 \\ 6 \\ 3 \end{pmatrix}$

In section 1.3, we saw that

$$\begin{vmatrix} 1 & 2 & 3 & 7 \\ -8 & 4 & 1 & 0 \\ 1 & 9 & 2 & 6 \\ -10 & 0 & 3 & 3 \end{vmatrix} \neq 0$$

and hence, by 3.1E, the four 4-dimensional vectors V_1, V_2, V_3, V_4 are linearly independent. By 3.2B, V_1, V_2, V_3, V_4 form a basis for a four dimensional vector space.

We may therefore express any four dimensional vector, V, as a linear combination of these four vectors. In particular, we may trivially express each of the vectors V_1, V_2, V_3, V_4, in the basis as a linear combination of the basis vectors.

Example:

$$V_3 = 0\ V_1 + 0\ V_2 + 1\ V_3 + 0\ V_4$$

i.e.

$$\begin{pmatrix} 3 \\ 1 \\ 2 \\ 3 \end{pmatrix} = 0 \begin{pmatrix} 1 \\ -8 \\ 1 \\ -10 \end{pmatrix} + 0 \begin{pmatrix} 2 \\ 4 \\ 9 \\ 0 \end{pmatrix} + 1 \begin{pmatrix} 3 \\ 1 \\ 2 \\ 3 \end{pmatrix} + 0 \begin{pmatrix} 7 \\ 0 \\ 6 \\ 3 \end{pmatrix}$$

CHAPTER 4: CONVEX SETS

4.1 Basic Notions

Let $X_1, X_2, \ldots, X_k$ be k n-dimensional vectors. By a <u>convex combination</u> of these vectors, we shall mean a linear combination:

$$a_1 X_1 + a_2 X_2 + \ldots + a_k X_k$$

where each a_i is a nonnegative real number (i.e. $a_i \geq 0$), and such that:

$$\sum_{i=1}^{k} a_i = a_1 + a_2 + \ldots + a_k = 1.$$

Example:

$$1/10 \begin{pmatrix} 3 \\ 8 \\ -2 \\ 1 \end{pmatrix} + 2/5 \begin{pmatrix} 1 \\ 4 \\ -5 \\ 0 \end{pmatrix} + 1/2 \begin{pmatrix} 0 \\ 7 \\ 3 \\ -2 \end{pmatrix}$$

) a convex combination of $\begin{pmatrix} 3 \\ 8 \\ -2 \\ 1 \end{pmatrix}$, $\begin{pmatrix} 1 \\ 4 \\ -5 \\ 0 \end{pmatrix}$, and $\begin{pmatrix} 0 \\ 7 \\ 3 \\ -2 \end{pmatrix}$ since $1/10 \geq 0$, $2/5 \geq 0$, $1/2 \geq 0$ such that $1/10 + 2/5 + 1/2 = 1$.

A set of n-dimensional vectors will be called <u>convex</u> if, and only if, for all pairs of vectors, X_1 and X_2, in this set, any convex combination, $a_1 X_1 + a_2 X_2$, is also in the set.

It can be shown (Gass, pg 28) that if $X_1, X_2, \ldots, X_k$ are any vectors in a convex set, then every convex combination, $a_1 X_1 + a_2 X_2 + \ldots + a_k X_k$, of these vectors is also in this convex set.

An extreme vector of a convex set of vectors is a vector which cannot be expressed as a convex combination of any other two distinct vectors in this set.

The above definitions appear to be unrelated to our objective and indeed to the material which preceded. The concepts of convex sets and extreme vectors will appear again, however, as a basic notion in Part II.

An interesting geometric interpretation can be given to convex combinations, convex sets, extreme vectors, etc. It is necessary, however, to think of a vector as a point in space. The reader may be able to adopt this frame of reference to treat the two dimensional vector $\begin{pmatrix}5\\3\end{pmatrix}$ as the point in the Cartesian plan whose coordinates are (5,3) or to treat the three dimensional vector $\begin{pmatrix}2\\-6\\5\end{pmatrix}$ as the point whose coordinates in three dimensional space are (2,-6, 5).

The uninitiated will have trouble, however, when trying to "see" a seven dimensional vector as a point in a seven dimensional space.

Nevertheless, a bit of insight may be gained from the discussion of convex sets of 2-dimensional vectors. The interested reader is referred to Gass, pp 28 and 29.

PART II: FUNDAMENTALS OF LINEAR PROGRAMMING

CHAPTER 1: STATEMENT OF THE PROBLEM

Consider a system of m linear equations in n unknowns where the number of unknowns or variables is greater than the number of equations (i.e., $n>m$).

Thus, our system is:

$$a_{11} X_1 + a_{12} X_2 + \ldots + a_{1n} X_n = b_1$$

$$a_{21} X_1 + a_{22} X_2 + \ldots + a_{2n} X_n = b_2$$

$$\vdots$$

$$a_{m1} X_1 + a_{m2} X_2 + \ldots + a_{mn} X_n = b_m$$

where the matrix of coefficients has more columns than rows.

We have seen, in examples (2) and (3) of Section 2.3, Part I, that such a system may have an infinite number of solutions. This will indeed be the case if the system has any solutions at all. That is:

A system of m equations in n unknowns where $n>m$ has either no solution or an infinite number of them. See Cullen, pp 1 and 55.

Recall from I-2.4A, that a system of simultaneous linear equations has a unique solution if, and only if, the number of variables is equal to the number of equations and $|A| \neq 0$ where A is the square matrix of coefficients.

Let us now return to our system of m equations in n unknowns where $\underline{n \geq m}$. Suppose that there is $\underline{a}$ solution to the system. Then, as we have stated, there will be an infinite number of solutions.

Let us now consider those solutions $X = \begin{pmatrix} x_1 \\ x_2 \\ x_n \end{pmatrix}$ to our system for which $\underline{X \geq 0} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$. By this, of course, we mean that $x_1 \geq 0$, $x_2 \geq 0$, . . . , $x_n \geq 0$.

We may now be considering fewer solutions than we were originally, but the number of such solutions may still be infinite.

Consider the expression:

$$c_1 x_1 + c_2 x_2 + \ldots + c_n x_n.$$

where the c_i are fixed. We shall call this expression our <u>objective function</u>.

We will obtain a value for this objective function everytime we substitute values for $x_1, x_2, \ldots, x_n$.

A <u>linear programming problem (or LP problem)</u> has the following form: Find the solution vector $X = \begin{pmatrix} x_1 \\ x_2 \\ x_n \end{pmatrix}$ which maximizes (or minimizes) our objective function:

$$c_1 x_1 + c_2 x_2 + \ldots + c_n x_n$$

subject to the following two conditions:

(1) $X = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}$ is a solution vector of the system:

$$a_{11}\ x_1 + a_{12}\ x_2 + \ .\ .\ .\ + a_{1n}\ x_n = b_1$$

$$a_{21}\ x_1 + a_{22}\ x_2 + \ .\ .\ .\ + a_{2n}\ x_n = b_2$$

$$\vdots$$

$$a_{m1}\ x_1 + a_{m2}\ x_2 + \ .\ .\ .\ + a_{mn}\ x_n = b_m$$

(2) $x_1 \geq 0,\ x_2 \geq 0,\ .\ .\ .\ .\ ,\ x_n \geq 0.$

The vector X which satisfies conditions (1) and (2) and for which the objective function is maximized (minimized) is called the (an) <u>optimal solution to the LP problem</u>.

Examples:

(1) Maximize $3x_1 + 2x_2 - 7x_3 + x_4$ subject to the restraints:

(1)
$$\begin{aligned} 4x_1 - x_2 + x_3 + 3x_4 &= 10 \\ x_1 + 5x_2 \phantom{{}+ x_3} + x_4 &= 3 \\ 3x_1 + x_2 + 2x_3 - x_4 &= 1 \end{aligned}$$

(2) $x_i \geq 0$; $i = 1, 2, 3, 4$

Our objective function here is

$$3x_1 + 2x_2 - 7x_3 + x_4$$

which we wish to maximize in this case.

(2) Minimize $1.3x_1 + 2.7x_2 - .4x_3$ subject to the restraints:

(1)
$$\begin{aligned} x_1 + x_2 + x_3 &\leq 20 \\ 2x_1 - x_2 + 4x_3 &\leq 50 \\ x_1 - x_2 - x_3 &\leq 10 \end{aligned}$$

Our objective here is

$$1.3x_1 + 2.7x_2 - .4x_3$$

which is to minimized in this case.

However, at first glance, condition (1) does not resemble a system of simultaneous linear equations. Let us consider this further. By the inequality $a \leq 4$, we mean that there exists some nonnegative real number (say b) such that a is "b less than 4." That is, $4 - b = a$ or $a + b = 4$.

Thus, from the inequality $x_1 + x_2 + x_3 \leq 20$, we obtain the implied equation:

$$x_1 + x_2 + x_3 + x_4 = 20$$

where x_4 is the nonnegative real number such that $x_4 = 20 - (x_1 + x_2 + x_3)$.

Obtaining similar equations for the second two inequalities, we have the following LP problem:

Minimize $1.3x + 2.7x_2 - .4x_3 + 0x_4 + 0x_5 + 0x_6$ subject to the restraints

$$\text{(1)} \quad \begin{aligned} x_1 + x_2 + x_3 + x_4 &= 20 \\ 2x_1 - x_2 + 4x_3 + x_5 &= 50 \\ x_1 - x_2 - x_3 + x_6 &= 10 \end{aligned}$$

$$\text{(2)} \quad x_i \quad 0 \quad ; \quad i = 1, 2, 3, 4, 5, 6$$

We call x_4, x_5, and x_6 slack variables.

It may occur that a slack variable assumes a nonzero value in the optimal solution. The significance of such an outcome will be discussed later.

We have seen that no problem arises if we have a system of inequalities since this system can be changed to a system of equations by introducing slack variables.

In the above example, however, we dealt with changing an inequality of the form: $a \leq 4$ to a equation $a + b = 4$ where $b \geq 0$. If we are faced with an inequality of the form $a \geq 4$, we merely write $-a \leq -4$ and obtain the equation $-a + b = -4$. In this case again, $b \geq 0$.

Consider again the general LP problem:

Maximize (minimize) the objective function

$$c_1 x_1 + c_2 x_2 + c_3 x_3 + \ldots + c_n x_n$$

subject to the restraints:

$$
\begin{aligned}
(1) \quad & a_{11} x_1 + a_{12} x_2 + \ldots + a_{1n} x_n = b_1 \\
& a_{21} x_1 + a_{22} x_2 + \ldots + a_{2n} x_n = b_2 \\
& \vdots \\
& a_{m1} x_1 + a_{m2} x_2 + \ldots + a_{mn} x_n = b_m
\end{aligned}
$$

$$(2) \quad x_i \geq 0 \; ; \quad i = 1, 2, 3, \ldots, n$$

Vector $X = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}$ which satisfies conditions (1) and (2) is called a

<u>feasible solution</u> to the LP problem.

With reference to the notation introduced in Section 2.4, Part I, we may state the above general LP problem as follows.

Maximize (minimize) the objective function

$$(c_1, c_2 , \ldots , c_n) \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}$$

subject to the restraints:

$$(1)\quad \begin{pmatrix} a_{11} & a_{12} & \cdot\ \cdot\ \cdot\ \cdot & a_{1n} \\ a_{21} & a_{22} & \cdot\ \cdot\ \cdot\ \cdot & a_{2n} \\ \vdots & & & \\ a_{m1} & a_{m2} & \cdot\ \cdot\ \cdot\ \cdot & a_{mn} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{pmatrix}$$

$$(2)\quad x_i \quad 0; \quad i = 1, 2, \ldots, n$$

In a more abbreviated form we can write:

Maximize (minimize) the objective function

$$CX$$

subject to the restraints:

$$(1)\quad AX = B$$

$$(2)\quad x \geq 0$$

where C is the 1 x n matrix $(c_1 k\ c_2, \ldots, c_n)$; X is the n x 1 matrix $\begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}$; A is the matrix of coefficients; B is the column of constants; and 0 is the n x 1 matrix $\begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$.

CHAPTER 2: MATHEMATICAL MODELS FOR LP PROBLEMS

Let us now turn our attention to examples of physical phenomena which give rise to LP problems. Of the following five examples, the first four were taken from Hellier, Lieberman, pp 129-135; the last from UP-4138, Appendix A.

2.1 The Product Manufacturing Problem

A manufacturing firm has discontinued production of a certain unprofitable product line. This created considerable excess production capacity. Management is considering devoting this excess capacity to one or more of three products; call them products 1, 2, and 3. The available capacity on the machines which might limit output is summarized in the following table.

Machine Type	Available time (in machine hours per week)
Milling Machine	200
Lathe	100
Grinder	50

The number of machine hours required for each unit of the respective products is given below.

Productivity
(in machine hours per unit)

Machine Type	Product 1	Product 2	Product 3
Milling Machine	8	2	3
Lathe	4	3	
Grinder	2		1

The sales department indicates that the sales potential for products 1 and 2 exceeds the maximum production rate and that the sales potential for product 3 is 20 units per week.

The unit profit would be \$20, \$6, and \$8, respectively, on products 1, 2, and 3.

The problem is to formulate a linear programming model for determining how much of each product the firm should produce in order to maximize profit. Let us now proceed to construct this model.

Let x_i $(i = 1, 2, 3)$ be the number of units of product i produced per week. Since profit has been chosen as the measure of effectiveness, the object is to maximize

$$20x_1 + 6x_2 + 8x_3,$$

subject to the restrictions developed below.

The "limited resources" in this situation are the available capacity of the three machine groups and the sales potential for product 3.

Therefore, a mathematical constraint must be developed to describe each of these resource restrictions. The first restriction is that no more than 200 milling machine hours per week can be allocated to the activities, the production of the three products. The number of milling machine hours actually allocated is $8x_1 + 2x_2 + 3x_3$. Therefore, the mathematical statement of the first restriction is

$$8x_1 + 2x_2 + 3x_3 \leq 200.$$

Similarly, the other two capacity restrictions are

$$4x_1 + 3x_2 \leq 100$$

$$2x_1 + x_3 \leq 50$$

The mathematical statement of the sales potential restriction obviously is

$$x_3 \leq 20$$

Finally, there are the nonnegativity restrictions.

Therefore, in summary, the linear programming model for this problem is the following:

Maximize $\quad 20x_1 + 6x_2 + 8x_3$

subject to the restraints:

$$\begin{array}{lrcrcrcl} (1) & 8x_1 & + & 2x_2 & + & 3x_3 & \leq & 200 \\ & 4x_1 & + & 3x_2 & & & \leq & 100 \\ & 2x_1 & & & + & x_3 & \leq & 50 \\ & & & & & x_3 & \leq & 20 \end{array}$$

$$(2) \quad x_1 \geq 0, \; x_2 \geq 0, \; x_3 \geq 0.$$

We shall see in Part III that to obtain a solution to this problem through the use of an electronic computer, we may leave the model in the above form. However, if we wish to calculate the solution by the simple method to be discussed in Chapter 4 of Part II, we must introduce the slack variables discussed in Chapter 1, Part II. We obtain:

Maximize $20x_1 + 6x_2 + 8x_3 + 0x_4 + 0x_5 + 0x_6 + 0x_7$

subject to the restraints:

$$
\begin{array}{llcl}
(1) & 8x_1 + 2x_2 + 3x_3 + x_4 & & = 200 \\
 & 4x_1 + 3x_2 \qquad\qquad + x_5 & & = 100 \\
 & 2x_1 \qquad + x_3 \qquad\qquad + x_6 & & = 50 \\
 & \qquad\qquad x_3 \qquad\qquad\qquad + x_7 & & = 20
\end{array}
$$

$$(2) \quad x_1 \geq 0,\ x_2 \geq 0,\ x_3 \geq 0,\ x_4 \geq 0,\ x_5 \geq 0,\ x_6 \geq 0,\ x_7 \geq 0.$$

Note that if x_4 takes on a positive value (rather than 0) in the optimal solution, the implication in that x_4 hours per week out of the 200 hours available on the milling machine should not be used.

2.2 The Diet Problem

One of the classic problems of linear programming is the diet problem. The objective is to ascertain the quantities of certain foods that should be eaten to meet certain nutritional requirements at a minimum cost. Assume that consideration is limited to milk, beef, and eggs, and to vitamins A, C, and D. Suppose that the number of milligrams of each of these vitamins contained within a unit of each food is as given below.

Vitamin	Gallon of Milk	Pound of beef	Dozen of eggs	Minimum Daily Requirements
A	1	1	10	1 mg.
C	100	10	10	50 mg.
D	10	100	10	10 mg.
Cost	\$1.00	\$1.10	\$0.50	

We proceed to construct the mathematical model for this LP problem.

Let x_1 = the number of gallons of milk in the daily diet

x_2 = the number of pounds of beef in the daily diet

x_3 = the number of dozens of eggs in the daily diet

The objective is to minimize cost, and the resource restrictions are in the form of lower bounds rather than upper bounds. Therefore, the LP model for this problem is the following:

Minimize $1.0x_1 + 1.1x_2 + 0.5x_3$

subject to the restrictions:

$$\begin{aligned} (1) \quad x_1 + x_2 + 10x_3 &\geq 1 \\ 100x_1 + 10x_2 + 10x_3 &\geq 50 \\ 10x_1 + 100x_2 + 10x_3 &\geq 10 \end{aligned}$$

$$(2) \quad x_i \geq 0; \quad i = 1, 2, 3$$

Introducing slack variables to put this model in the form previously discussed, we have:

Minimize $1.0x_1 + 1.1x_2 + 0.5x_3 + 0x_4 + 0x_5 + 0x_6$

subject to the restrictions:

$$
\begin{aligned}
(1)\qquad -x_1 - x_2 - x_3 + x_4 &= -1 \\
-100X_1 - 10x_2 - 10x_3 + x_5 &= -50 \\
-10X_1 - 100x_2 - 10x_3 + x_6 &= -10
\end{aligned}
$$

$$(2)\qquad x_1 \geq 0,\ x_2 \geq 0,\ x_3 \geq 0,\ x_4 \geq 0,\ x_5 \geq 0,\ x_6 \geq 0$$

We repeat, however, that it is not necessary to put the model in this final form when utilizing an electronic computer. This will be discussed in Part III.

2.3 The Gasoline Mix Problem

Consider a product mix problem within the context of a simplified oil refinery situation. Suppose that the refinery wishes to blend four petroleum constituents into three grades of gasoline, A, B, and C. The problem is to determine the mix of the four constituents given below.

Constituent	Maximum quantity available in barrels per day	Cost per barrel
1	3,000	\$3
2	2,000	\$6
3	4,000	\$4
4	1,000	\$5

To maintain the required quality for each grade of gasoline, it is necessary to specify certain maximum or minimum percentages of the constituents in each blend. These are given below, along with the selling price for each grade.

Grade	Specification	Selling price per barrel
A	Not more than 30% of 1	
	Not less than 40% of 2	
	Not more than 50% of 3	$5.50
	Not more than 50% of 1	
B	Not less than 10% of 2	$4.50
C	Not more than 70% of 1	$3.50

Assume that the "profit" to be maximized is total sales income minus the total cost of the constituents.

We proceed to construct a mathematical model for this problem. Let Y_{ij} (i = A, B, C; j = 1, 2, 3, 4) be the total number of barrels of constituent j allocated to gasoline grade i per day. That is:

Y_{A1} = the number of barrels of constituent 1 allocated to gas. gr. A per day

Y_{A2} = " " " " " " " " " 2 " " " " " A " " "

'

'

Y_{B3} = " " " " " " " " " 3 " " " " " B " "

'

'

'

Y_{C4} = " " " " " " " " " 4 " " " " " C " "

The total amount of gasoline grade i produced per day is then

$$y_{i1} + y_{i2} + y_{i3} + y_{i4}$$

The proportion of constituent j in gasoline grade i is

$$\frac{y_{ij}}{y_{i1} + y_{i2} + y_{i3} + y_{i4}}$$

The total profit is given by

$$5.5\ (Y_{A1} + Y_{A2} + Y_{A3} + Y_{A4}) + 4.5\ (Y_{B1} + Y_{B2} + Y_{B3} + Y_{B4})$$
$$+3.5\ (Y_{c1} + Y_{c2} + Y_{c3} + Y_{c4}) - 3\ (Y_{A1} + Y_{B1} + Y_{C1})$$
$$-6\ (Y_{A2} + Y_{B2} + Y_{C2}) -4\ (Y_{A3} + Y_{B3} + Y_{C3}) - 5\ (Y_{A4} + Y_{B4} + Y_{C4}).$$

which, when like terms are combined, becomes

$$2.5\ Y_{A1} -0.5\ Y_{A2} + 1.5\ Y_{A3} + 0.5\ Y_{Ar} + 1.5\ Y_{B1} - 1.5\ Y_{B2} + 0.5\ Y_{B3}$$
$$-0.5\ Y_{B4} + 0.5\ Y_{C1} - 2.5\ Y_{C2} - 0.5\ Y_{C3} - 1.5\ Y_{C4}$$

We must therefore maximize this profit function (our objective function) subject to the restrictions imposed by the <u>availability of constituents</u>, the <u>blend requirements</u>, and by the requirement that $y_{ij} \geq 0$ for $i = A, B, C;\ j = 1, 2, 3, 4$.

The availability restrictions clearly are:

$$Y_{A1} + Y_{B1} + Y_{C1} \leq 3000$$
$$Y_{A2} + Y_{B2} + Y_{C2} \leq 2000$$
$$Y_{A33} + Y_{B3} + Y_{C3} \leq 4000$$
$$Y_{A4} + Y_{B4} + Y_{C4} \leq 1000$$

The blend restrictions for gasoline grade A are:

$$Y_{A1} \leq 0.3\ (Y_{A1} + Y_{A2} + Y_{A3} + Y_{A4})$$
$$Y_{A2} \geq 0.4\ (Y_{A1} + Y_{A2} + Y_{A3} + Y_{A4})$$
$$Y_{A3} \leq 0.5\ (Y_{A1} + Y_{A2} + Y_{A3} + Y_{A4})$$

However, these restrictions are not in a convenient form for a linear programming model, so they should be rewritten as:

$$0.7\ Y_{A1} - 0.3\ Y_{A2} - 0.3\ Y_{A3} - 0.3\ Y_{A4} \leq 0$$

$$-0.4\ Y_{A1} + 0.6\ Y_{A2} - 0.4\ Y_{A3} - 0.4\ Y_{A4} \geq 0$$

$$-0.5\ Y_{A1} - 0.5\ Y_{A2} + 0.5\ Y_{A3} - 0.5\ Y_{A4} \leq 0$$

Similarly, the final forms of the blend restrictions for gasoline grades B and C are:

$$0.5\ Y_{B1} - 0.5\ Y_{B2} - 0.5\ Y_{B3} - 0.5\ Y_{B4} \leq 0$$

$$-0.1\ Y_{B1} + 0.9\ Y_{B2} - 0.1\ Y_{B3} - 0.1\ Y_{B4} \geq 0$$

$$0.3\ Y_{C1} - 0.7\ Y_{C1} - 0.7\ Y_{C3} - 0.7\ Y_{C4} \leq 0$$

Let us now make the following substitutions in order that our model resembles the previous two examples.

Let:

$$x_1 = y_{A1}$$
$$x_2 = y_{A2}$$
$$x_3 = y_{A3}$$
$$x_4 = y_{A4}$$
$$x_5 = y_{B1}$$
$$x_6 = y_{B2}$$
$$x_7 = y_{B3}$$
$$x_8 = y_{B4}$$
$$x_9 = y_{C1}$$
$$x_{10} = y_{C2}$$
$$x_{11} = y_{C3}$$
$$x_{12} = y_{C4}$$

The model for our LP problem then has the form:

$$\text{Maximize}\quad 2.5x_1 -0.5x_2 +1.5x_3 +0.5x_4 +1.5x_5 -1.5x_6 +0.5x_7 -0.5x_8 +0.5x_9 -2.5x_{10} -0.5x_{11} -1.5x_{12}$$

subject to:

(1)

$$\begin{array}{l}
x_1 + x_5 + x_9 \leq 3000 \\
x_2 + x_6 + x_{10} \leq 2000 \\
x_3 + x_7 + x_{11} \leq 4000 \\
x_4 + x_8 + x_{12} \leq 1000 \\
0.7x_1 -0.3x_2 -0.3x_3 -0.3x_4 \leq 0 \\
-0.4x_1 +0.6x_2 -0.4x_3 -0.4x_4 \geq 0 \\
-0.5x_1 -0.5x_2 +0.5x_3 -0.5x_4 \leq 0 \\
0.5x_5 -0.5x_6 -0.5x_7 -0.5x_8 \leq 0 \\
-0.1x_5 +0.9x_6 -0.1x_7 -0.1x_8 \geq 0 \\
0.3x_9 -0.7x_{10} -0.7x_{11} -0.7x_{12} \leq 0
\end{array}$$

(2) $x_i \geq 0; \quad i = 1, 2, \ldots, 12$

Introducing slack variables, our model becomes:

$$\text{Maximize}\quad 2.5x_1 -0.5x_2 -1.5x_3 +0.5x_4 +1.5x_5 -1.5x_6 +0.5x_7 -0.5x_8 +0.5x_9 -2.5x_{10} -0.5x_{11} -1.5x_{12} +0x_{13} +0x_{14} +0x_{15} +0x_{16} +0x_{17} +0x_{18} +0x_{19} +0x_{20} +0x_{21} +0x_{22}$$

subject to the restrictions:

(1)

$$
\begin{array}{llllllllllllllllllllll}
x_1 & & & & +x_5 & & & & +x_9 & & & & x_{13} & & & & & & & & = & 3000 \\
& x_2 & & & & +x_6 & & & & +x_{10} & & & +x_{14} & & & & & & & & = & 2000 \\
& & x_3 & & & & +x_7 & & & & +x_{11} & & & +x_{15} & & & & & & & = & 4000 \\
& & & x_4 & & & & +x_8 & & & & +x_{12} & & & +x_{16} & & & & & & = & 1000 \\
0.7x_1 & -0.3x_2 & -0.3x_3 & -0.3x_4 & & & & & & & & & & & & +x_{17} & & & & & = & 0 \\
-0.4x_1 & -0.6x_2 & +0.4x_3 & +0.4x_4 & & & & & & & & & & & & & +x_{18} & & & & = & 0 \\
-0.5x_1 & -0.5x_2 & +0.5x_3 & -0.5x_4 & & & & & & & & & & & & & & +x_{19} & & & = & 0 \\
& & & & 0.5x_5 & -0.5x_6 & -0.5x_7 & -0.5x_8 & & & & & & & & & & & +x_{20} & & = & 0 \\
& & & & +0.1x_5 & -0.9x_6 & +0.1x_7 & +0.1x_8 & & & & & & & & & & & & +x_{21} & = & 0 \\
& & & & & & & & 0.3x_9 & -0.7x_{10} & -0.7x_{11} & -0.7x_{12} & & & & & & & & +x_{22} & = & 0
\end{array}
$$

2.4 The Farming Problem

A certain farming organization operates three farms of comparable productivity. The output of each farm is limited both by the usable acreage and by the amount of water available for irrigation. The data for the upcoming season are the following.

Farm	Usable Acreage	Water Available (in pure feet)
1	400	1500
2	600	2000
3	300	900

The organization is considering three crops for planting which differ primarily in their expected profit per acre and in their consumption of water. Furthermore, the total acreage that can be devoted to each of the crops is limited by the amount of appropriate harvesting equipment available.

Crop	Maximum Acreage	Water consumption in acre feet per acre	Expected profit per acre
A	700	5	$400
B	800	4	$300
C	300	3	$100

In order to maintain a uniform workload among the farms, it is the policy of the organization that the percentage of the usable acreage planted must be the same at each farm. However, any combination of the crops may be grown at any of the farms. The organization wishes to know how much of each crop should be planted at the respective farms in order to maximize expected profit.

Let us now proceed to construct a mathematical model of this LP problem. Let y_{ij} = the number of acres at the i-th farm devoted to the j-th crop.

$$(i = 1, 2, 3; \quad j = A, B, C)$$

That is:

y_{1A} = the number of acres at farm 1 devoted to crop A

y_{2A} = the number of acres at farm 2 devoted to crop A

'

'

y_{3B} = the number of acres at farm 3 devoted to crop B

'

'

y_{4C} = the number of acres at farm 4 devoted to crop C

The objective function (i.e., the profit function) is therefore given by:

$$400(y_{1A} + y_{2A} + y_{3A}) + 300(y_{1B} + y_{2B} + y_{3B}) + 100(y_{1C} + y_{2C} + y_{3C})$$

The restrictions on usable acreage at each farm are

$$y_{1A} + y_{1B} + y_{1C} \leq 400$$
$$y_{2A} + y_{2B} + y_{2C} \leq 600$$
$$y_{3A} + y_{3B} + y_{3C} \leq 300$$

The restrictions on water availability are

$$5y_{1A} + 4y_{1B} + 3y_{1C} \leq 1500$$
$$5y_{2A} + 4y_{2B} + 3y_{2C} \leq 2000$$
$$5y_{3A} + 4y_{3B} + 3y_{3C} \leq 900$$

The crop restrictions on acreage are

$$y_{1A} + y_{2A} + y_{3A} \leq 700$$
$$y_{1B} + y_{2B} + y_{3B} \leq 800$$
$$y_{1C} + y_{2C} + y_{3C} \leq 300$$

Because of the policy of a uniform workload, the equations,

$$\frac{y_{1A} + y_{1B} + y_{1C}}{400} = \frac{y_{2A} + y_{2B} + y_{2C}}{600}$$

$$\frac{y_{2A} + y_{2B} + y_{2C}}{600} = \frac{y_{3A} + y_{3B} + y_{3C}}{300}$$

$$\frac{y_{1A} + y_{1B} + y_{1C}}{400} = \frac{y_{3A} + y_{3B} + y_{3C}}{300}$$

must be satisfied. Since the first two equations imply the third, the third equation may be omitted from the model. Furthermore, these equations are not yet in a convenient form for a linear programming model since all of the variables are not on the left-hand side. Hence, the final forms of the uniform workload restrictions are

$$3(y_{1A} + y_{1B} + y_{1C}) - 2(y_{2A} + y_{2B} + y_{2C}) = 0$$
$$y_{2A} + y_{2B} + y_{2C} - 2(y_{3A} + y_{3B} + y_{3C}) = 0$$

Following the procedure of the previous problem, we let

$$x_1 = y_{1A}$$

$$x_2 = y_{2A}$$

$$x_3 = y_{3A}$$

$$x_4 = y_{1B}$$

$$x_5 = y_{2B}$$
$$x_6 = y_{3B}$$
$$x_7 = y_{1C}$$
$$x_8 = y_{2C}$$
$$x_9 = y_{3C}$$

Our model for this problem then becomes:

Maximize $400(x_1 + x_2 + x_3) + 300(x_4 + x_5 + x_6) + 100(x_7 + x_8 + x_9)$ subject to the restrictions:

(1)

$$\begin{array}{ccccccccccc}
x_1 & & & +x_4 & & & +x_7 & & & \leq & 400 \\
 & x_2 & & & +x_5 & & & +x_8 & & \leq & 600 \\
 & & x_3 & & & +x_6 & & & +x_9 & \leq & 300 \\
5x_1 & & & +4x_4 & & & +3x_7 & & & \leq & 1500 \\
 & 5x_2 & & & +4x_5 & & & +3x_8 & & \leq & 2000 \\
 & & 5x_3 & & & +4x_6 & & & +3x_9 & \leq & 900 \\
x_1 & +x_2 & +x_3 & & & & & & & \leq & 700 \\
 & & & x_4 & +x_5 & +x_6 & & & & \leq & 800 \\
 & & & & & & x_7 & +x_8 & +x_9 & \leq & 300 \\
3x_1 & -2x_2 & & -3x_4 & -2x_5 & & +3x_7 & -2x_8 & & = & 0 \\
 & x_2 & -2x_3 & & +x_5 & -2x_6 & & +x_8 & -2x_9 & = & 0
\end{array}$$

(2) $x_i \geq 0; \quad i = 1, 2, \ldots, 9$

Introducing slack variables, our model becomes:

Maximize $400(x_1 + x_2 + x_3) + 300(x_4 + x_5 + x_6) + 100(x_7 + x_8 + x_9)$
$+0x_{10} + 0x_{11} + 0x_{12} + 0x_{13} + 0x_{14} + 0x_{15} + 0x_{16} + 0x_{17} + 0x_{18}$

subject to the restrictions:

(1)

$$
\begin{array}{llllllllllllllllllll}
x_1 & & & +x_4 & & & +x_7 & & & +x_{10} & & & & & & & & & = & 400 \\
 & x_2 & & & +x_5 & & & +x_8 & & & +x_{11} & & & & & & & & = & 600 \\
 & & x_3 & & & +x_6 & & & +x_9 & & & +x_{12} & & & & & & & = & 300 \\
5x_1 & & & +4x_4 & & & +3x_7 & & & & & & +x_{13} & & & & & & = & 1500 \\
 & 5x_2 & & & +4x_5 & & & +3x_8 & & & & & & +x_{14} & & & & & = & 2000 \\
 & & 5x_3 & & & +4x_6 & & & +3x_9 & & & & & & +x_{15} & & & & = & 900 \\
x_1 & +x_2 & +x_3 & & & & & & & & & & & & & +x_{16} & & & = & 700 \\
 & & & x_4 & +x_5 & +x_6 & & & & & & & & & & & +x_{17} & & = & 800 \\
 & & & & & & x_7 & +x_8 & +x_9 & & & & & & & & & +x_{18} & = & 300 \\
3x_1 & -2x_2 & & +3x_4 & -2x_5 & & +3x_7 & -2x_8 & & & & & & & & & & & = & 0 \\
 & x_2 & -2x_3 & & +x_5 & -2x_6 & & +x_8 & -2x_9 & & & & & & & & & & = & 0
\end{array}
$$

(2) $x_i \geq 0; \quad i = 1, 2, \ldots, 18$

2.5 The Nut Mix Problem

A manufacturer wishes to determine an optimal program for mixing three grades of nuts consisting of cashews, hazels, and peanuts according to the specifications and prices listed below. Hazels may be introduced into the mixture in any quantity, provided the specifications below are met.

Mixture	Specifications	Selling price $/lb.
A	Not less than 50% cashews	
	Not more than 25% peanuts	.50
B	Not less than 25% cashews	
	Not more than 50% peanuts	.35
D	No specifications	.25

Now, suppose that the manufacturer has certain capacity limits on the amounts of inputs he can employ. Let these limitations and the price of the inputs appear as follows:

Inputs	Capacity (lb./day)	Price ($/lb.)
Cashews	100	.65
Peanuts	100	.25
Hazels	60	.35
Total	260	

The manufacturer wishes to determine the quantity of each type of nut in each of the three mixtures in order to maximize profit.

We proceed to construct a mathematical model of this LP problem. We deviate slightly from the notation of the 1108 Linear Programming User's Manual in order to conform with our previous examples.

Let C, P, and H represent Cashews, Peanuts, and Hazels respectively. Next let y_{ij} = the number of nuts of type j in mixture i. (i = A, B, D; j = C, P, H). i.e. y_{AC} = the number of Cashews in mixture A, etc.

The total number of nuts in mixture A is then given by

$$y_{AC} + y_{AP} + y_{AH}$$

Likewise, the total number of nuts in mixture B is

$$y_{BC} + y_{BP} + y_{BH}$$

and the total number of nuts in mixture D is

$$y_{DC} + y_{DP} + y_{DH}$$

On the other hand, the total number of cashews in all three mixtures is given by

$$y_{AC} + y_{BC} + y_{DC}$$

Likewise, the total number of peanuts in the three mixtures is

$$y_{AP} + y_{BP} + y_{DP}$$

and the total number of hazels in the three mixtures is

$$y_{AH} + y_{BH} + y_{DH}$$

Hence, our mixture requirements are given by

$$y_{AC} \geq 1/2(y_{AC} + y_{AP} + y_{AH})$$

$$y_{AP} \leq 1/4(y_{AC} + y_{AP} + y_{AH})$$

$$y_{BC} \geq 1/4(y_{BC} + y_{BP} + y_{BH})$$

$$y_{BP} \leq 1/2(y_{BC} + y_{BP} + y_{BH})$$

or, rewriting these inequalities to have all variables on the same side of the order relation, we have

$$-1/2\ y_{AC} + 1/2\ y_{AP} + 1/2\ y_{AH} \leq 0$$

$$-1/4\ y_{AC} + 3/4\ y_{AP} - 1/4\ y_{AH} \leq 0$$

$$-3/4\ y_{BC} + 1/4\ y_{BP} + 1/4\ y_{BH} \leq 0$$

$$-1/2\ y_{BC} + 1/2\ y_{BP} - 1/2\ y_{BH} \leq 0$$

Our capacity constraints are given by

$$y_{AC} + y_{BC} + y_{DC} \leq 100$$

$$y_{AP} + y_{BP} + y_{DP} \leq 100$$

$$y_{AH} + y_{BH} + y_{DH} \leq 60$$

Our objective function (profit function) is given by

$$.50(y_{AC} + y_{AP} + y_{AH}) + .35(y_{BC} + y_{BP} + y_{BH}) + .25(y_{DC} + y_{DP} + y_{DH})$$
$$-.65(y_{AC} + y_{BC} + y_{DC}) - .25(y_{AP} + y_{BP} + y_{DP}) - .35(y_{AH} + y_{BH} + y_{DH})$$

Following the convention of the previous two examples, we adopt the following substitutions:

$$x_1 = y_{AC}$$
$$x_2 = y_{AP}$$
$$x_3 = y_{AH}$$
$$x_4 = y_{BC}$$
$$x_5 = y_{BP}$$
$$x_6 = y_{BH}$$
$$x_7 = y_{DC}$$
$$x_8 = y_{DP}$$
$$x_9 = y_{DH}$$

The mathematical model for this LP problem is then:

Maximize

$$.50(x_1 + x_2 + x_3) + .35(x_4 + x_5 + x_6) + .25(x_7 + x_8 + x_9) - .65(x_1 + x_4 + x_7)$$
$$-.25(x_2 + x_5 + x_8) - .35(x_3 + x_6 + x_9)$$

$$= \text{(after combining terms)} \quad -.15x_1 + .25x_2 + .15x_3 - .30x_4 + .10x_5 + +0x_6 - .40x_7 + 0x_8 - .10x_9$$

subject to the restraints:

(1)

$$
\begin{array}{lllllllllll}
-0.5\,x_1 & + 0.5\,x_2 & + 0.5\,x_3 & & & & & & & \leq & 0 \\
-0.25x_1 & + 0.75x_2 & - 0.25x_3 & & & & & & & \leq & 0 \\
 & & & -0.75x_4 & + 0.25x_5 & + 0.25x_6 & & & & \leq & 0 \\
 & & & -0.5\,x_4 & + 0.5\,x_5 & - 0.5\,x_6 & & & & \leq & 0 \\
x_1 & & & +x_4 & & & +x_7 & & & \leq & 100 \\
 & x_2 & & & +x_5 & & & +x_8 & & \leq & 100 \\
 & & x_3 & & & +x_6 & & & +x_9 & \leq & 60
\end{array}
$$

(2) $x_i \geq 0; \quad i - 1, 2, \ldots, 9$

Introducing slack variables, our model becomes

Maximize $-.15x_1 + .25x_2 + .15x_3 - .30x_4 + .10x_5 + 0x_6 - .40x_7 +$
$0x_i - .10x_9 - 0x_{10} + 0x_{11} + 0x_{12} + 0x_{13} + 0x_{14} +$
$0x_{15} + 0x_{16}$

subject to the restraints:

(1)

$$
\begin{array}{rrrrrrrrrrrrrrrrl}
-0.5\,x_1 & +0.5\,x_2 & +0.5\,x_3 & & & & & & & x_{10} & & & & & & & = 0 \\
-0.25x_1 & +0.75x_2 & -0.25x_3 & & & & & & & & x_{11} & & & & & & = 0 \\
 & & & -0.75x_4 & +0.25x_5 & +0.25x_6 & & & & & & x_{12} & & & & & = 0 \\
 & & & -0.5\,x_4 & +0.5\,x_5 & -0.5\,x_6 & & & & & & & x_{13} & & & & = 0 \\
x_1 & & & +x_4 & & & +x_7 & & & & & & & x_{14} & & & = 100 \\
 & x_2 & & & +x_5 & & & +x_8 & & & & & & & x_{15} & & = 100 \\
 & & x_3 & & & +x_6 & & & +x_9 & & & & & & & x_{16} & = 60
\end{array}
$$

(2) $x_i \geq 0; \quad i = 1, 2, \ldots, 16$

CHAPTER 3: ANALYSIS OF FEASIBLE SOLUTIONS

3.1 Obtaining a Set of Feasible Solutions

In this and the next chapter, we shall assume that the objective function of our LP problem is to be <u>minimized</u>. We shall call attention to the points at which our statements differ from the case where the objective function is to be maximized when such variations arise.

Consider again the general LP problem:

Minimize $c_1 x_1 + c_2 x_2 + \ldots + c_n x_n$

subject to the restraints:

$$(1) \quad \begin{aligned} a_{11} x_1 + a_{12} x_2 + \ldots + a_{1n} x_n &= b_1 \\ a_{21} x_1 + a_{22} x_2 + \ldots + a_{2n} x_n &= b_2 \\ &\vdots \\ a_{m1} x_1 + a_{m2} x_2 + \ldots + a_{mn} x_n &= b_m \end{aligned}$$

$$(2) \quad x_i \geq 0, \ i = 1, 2, \ldots, n$$

or symbolically:

Minimized CX subject to:

(1) $AX = B$

(2) $X \geq 0$

Let us look at restraint (1), the system of m simultaneous linear equations in n unknowns.

Viewing this system as merely an abstract set of equations, unrelated to any physical phenomena, we can make no assumptions. However, if this system arises as the mathematical model of a valid LP problem based on some set of physical conditions, we may assume the following:

(1) The number of variables is greater that or equal to the number of equations (i.e. $n \geq m$).

(2) The rank of the m x n matrix of coefficients, A, is m.

(3) Any m columns of A, when thought of as m-dimensional vectors, are linearly independent. This amounts to saying that if D is any square m x m submatrix of A, then $|D| \neq 0$ and hence D^{-1} exists.

Suppose now that we wish to obtain a feasible solution to our LP problem. In the system,

$$
\begin{array}{l}
a_{11} x_1 + a_{10} x_2 + \cdots + a_{1n} x_n = b_1 \\
a_{21} x_1 + a_{22} x_2 + \cdots + a_{2n} x_n = b_2 \\
\vdots \\
a_{m1} x_1 + a_{m2} x_2 + \cdots + a_{mn} x_n = b_m
\end{array}
$$

we choose any m variables and let the remaining n-m variables be equal to zero.

Our system of m equations in n unknowns, AX = B, then reduces to a system of m equations in m unknowns, DX = B, where D is the square m x m matrix whose m columns are the columns of A which correspond to the m selected variables.

By assumption (3), $|D| \neq 0$, and hence D^{-1} exists. We may therefore solve for X' to obtain $X' = D^{-1}B$. Here, we have adopted the symbol X' to designate the m x 1 matrix of unknowns being solved for. Henceforth, we shall use X and X' interchangeably--determining whether we mean an m x 1 matrix or an n x 1 matrix by the context.

We will obtain a solution, X, to our system AX = B for each choice of D which we make.

The question arises: How many choices of D are there? Recall that D is determined by choosing m of the n variables. For each way of choosing these m variables, there corresponds a unique square matrix, D, whose construction is discussed above.

There will be as many matrices, D, then, as there are ways of choosing m of the n variables. If we were to choose two items from a set of four, we might choose the first and second, the second and third, the third and fourth, the first and third, the first and fourth, or the second and fourth. That is, from a set of four items, we can choose two items in six different ways.

In general, from a set of n items, we may choose m items in $\binom{m}{n}$ different ways. Here, the symbol $\binom{m}{n}$ represents the number

$$\frac{n!}{m!(n-m)!}, \text{ eg.} \binom{4}{2} = \frac{4!}{2!(4-2)!} = \frac{4!}{2!2!} = \frac{(4)\ (3)\ (2)\ (1)}{(2)\ (1)\ (2)\ (1)}$$

Thus, given the system AX = B which must be satisfied by a solution to our LP problem, we can obtain a solution (to the system

$AX = B$) by choosing <u>any</u> m of the n variables, letting the remaining n-m variables be equal to zero, and solving the resulting system of m equations in m unknowns, $DX = B$ to obtain $X = D^{-1}B$.

We will then have $\binom{n}{m}$ such solution vectors to our system $AX = B$. Any such solution vector, X, for which $X \geq 0$ (i.e. $x_i \geq 0$; $i = 1, 2, \ldots, n$) will also be a <u>feasible solution</u> to our LP problem.

We will not have obtained <u>all</u> of the solutions to our system $AX = B$ by this method of choosing m of the variables and letting the remaining n-m be equal to zero. For example, a solution to the system $AX = B$ which has fewer than n-m zero components will not be arrived at by the above procedure.

Keeping in mind, however, that our aim here is to find the "best" solution to the system $AX = B$ for which $X \geq 0$, we may not have to consider all of the solutions to the system $AX = B$. This will indeed be the case as we shall see in the next section.

3.2 Basic Theorems

Given the LP problem:

Minimize CX subject to:

(1) $AX = B$

(2) $X \geq 0$,

we recall that a <u>feasible solution</u> to this LP problem is a <u>vector</u> X which satisfies (1) and (2).

A <u>basic feasible solution</u> is a feasible solution with no more than m positive x_i. That is, at least n-m of the x_i are zero. We note that the feasible solutions which we obtained in 3.1 were basic feasible solutions.

An optimal solution is a feasible solution which minimizes, in this case, the objective function, CX.

We have the following useful theorems (Gass, pp 46, 47, and 52):

(1) The set of all feasible solutions to our LP problem is a convex set, K.

(2) The extreme vectors of this convex set, K, are merely the feasible solutions which we obtained in Section 3.1.

(3) If the objective function, CX, (i.e., $c_1 x_1 + c_2 x_2 + \dots + c_n x_n$) assumes an optimal value, it does so for an extreme vector of K. If CX assumes its optimal value for more than one extreme vector of K, then it takes on the same value for every convex combination of these vectors.

Let us now suppose that the LP problem:

Minimize CX subject to:

(1) $AX = B$

(2) $X \geq 0$

has an optimal solution.

We now have a method, although not a very efficient one, of finding this solution. From Section 3.1 and the above theorems, we see that our procedure is as follows.

(1) Choose m of the variables from $x_1, x_2, \dots, x_n$. Set the remaining n-m variables equal to zero.

This reduces the problem of finding a solution to the system $AX = B$ to the problem of finding the unique solution to $DX = B$ where D is as constructed in Section 3.1. This gives the unique solution $X = D^{-1}B$.

(2) We will obtain $\binom{n}{m}$ solution vectors, X, to our system $AX = B$ by step one. We must then check each of these $\binom{n}{m}$ solutions to see which satisfies $X \geq 0$. Such X will constitute a set of feasible solutions to our LP problem. In fact, the feasible solutions so obtained will constitute the extreme vectors of the convex set of all feasible solutions.

(3) Since we have assumed that our LP problem has an optimal solution, we know by Theorem 3 of this section that our LP problem will achieve this value for one of the extreme vectors of K.. (i.e., for one of the vectors obtained from step (2) above.) There will not be more than $\binom{n}{m}$ such vectors. We therefore evaluate CX for each of these vectors and designate the vector which produces the smallest value, in this case, of CX as the optimal solution to our LP problem.

Note that what we have arrived at is a method for arriving at an optimal solution for our LP problem <u>if</u> we know that an optimal solution exists.

The procedure is tedious and time consuming since we must analyze $\binom{n}{m}$ vectors. However, the method does contain a finite number of steps and can be carried out with a considerable amount of time and patience.

Since it is desirable to arrive at the optimal solution to an LP problem in the most efficient manner, the above procedure must be altered somewhat.

In general, the procedure is to start with a given extreme vector, to analyze it and then to obtain another extreme vector by changing only one of the components. We continue this--stopping at a "certain stage"--being assured that the final vector in this sequence is our optimal solution. The technique involved in this procedure is called the "Simplex Method" and is discussed in the next chapter.

CHAPTER 4: THE SIMPLEX METHOD

4.1 Introduction

We shall present merely the mechanics of the simplex method here. The interested reader with a sufficient mathematical background may find the theoretical justification for this technique interesting. Reference is made to Gass, pp 59-71.

Although our approach will be "cook-bookish" in nature, the fact that we imply the technique of putting a matrix in its row reduced echelon form--and call this procedure just that when we do use it--raises our discussion one level above the usual presentation of this technique.

Let us now look at the system

$$\begin{aligned}
a_{11} x_1 + a_{12} x_2 + \cdots + a_{1n} x_n &= b_1 \\
a_{21} x_1 + a_{22} x_2 + \cdots + a_{2n} x_n &= b_2 \\
&\vdots \\
a_{m1} x_1 + a_{m2} x_2 + \cdots + a_{mn} x_n &= b_m
\end{aligned}$$

Consider the matrix of coefficients

$$\begin{pmatrix}
a_{11} & a_{12} & \cdots & a_{1n} \\
a_{21} & a_{22} & \cdots & a_{2n} \\
\vdots & & & \\
a_{m1} & a_{m2} & \cdots & a_{mn}
\end{pmatrix}$$

By the vector P_1, we mean the first column of this matrix.

By the vector P_2, we mean the second column of this matrix, etc.

i.e. $P_1 = \begin{pmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{pmatrix}$; $P_2 = \begin{pmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{m2} \end{pmatrix}$; . . . $P_n = \begin{pmatrix} a_{1n} \\ a_{2n} \\ \vdots \\ a_{mn} \end{pmatrix}$

The vector P_i, of course, corresponds to the variable, x_i. We may therefore state assumption (3) of Section 3.1 as follows.

** Any collection of m vectors from the set $P_1, P_2, \ldots, P_n$ are linearly independent. * * *

With reference to section 3.2, we choose m of the n variables and set the remaining n-m equal to zero. Let us refer to the m variables which we choose as the <u>basis variables</u>. If x_i is a basis variable, we shall call the corresponding P_i <u>a basis vector</u>.

The matrix, D, which we constructed in Section 3.2, then, was merely m x m matrix whose columns were the basis vectors.

4.2 Procedure

Consider the LP problem:

Minimize $\quad x_2 - 3x_3 + 2x_5$

subject to the restraints:

$$\begin{array}{lrrrrrrl}
(1) & x_1 + & 3x_2 - & x_3 & & + 2x_5 & & = 7 \\
 & & -2x_2 + & 4x_3 + & x_4 & & & = 12 \\
 & & -4x_2 + & 3x_3 & & + 8x_5 & + x_6 & = 10
\end{array}$$

(2) $x_i \geq 0$; $i = 1, 2, \ldots, n$

We see that if we choose x_1, x_4, and x_6 as our basis variables (letting $x_2 = x_3 = x_5 = 0$), our system becomes

$$\begin{aligned} x_1 \qquad\qquad &= 7 \\ x_4 \qquad &= 12 \\ x_6 &= 10 \end{aligned}$$

Therefore, we can merely read the solution vector, $X = \begin{pmatrix} x_1 \\ x_4 \\ x_6 \end{pmatrix} = \begin{pmatrix} 7 \\ 12 \\ 10 \end{pmatrix}$

(or $X = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{pmatrix} = \begin{pmatrix} 7 \\ 0 \\ 0 \\ 12 \\ 0 \\ 10 \end{pmatrix}$ We need not construct D and D^{-1} in this case. However, to be formal, we may write the above as $DX = B$

or $IX = B$ since D in this case is the 3 x 3 identity matrix, identity matrix, $\begin{pmatrix} 1&0&0 \\ 0&1&0 \\ 0&0&1 \end{pmatrix}$ Thus, $X = I^{-1}B = B$ since $I^{-1} = I$.

It will occur in our procedure that we know which variables we wish to be our basis variables. It may therefore be advantageous to consider a system equivalent to our original but in which D, the matrix of basis vectors is the identity.

We illustrate this with our example. We have decided that x_1, x_4, and x_6 are to be our basis variables and hence P_1, P_4, P_6 are our basis vectors.

We therefore construct the following table.

P_1	P_4	P_6	P_2	P_3	P_5	P_0
1	0	0	3	-1	2	7
0	1	0	-2	4	0	12
0	0	1	-4	3	8	10

where P_0 is merely the column of constants. The 3 x 7 augmented matrix corresponds to the system:

$$\begin{array}{lllll} x_1 & & & + 3x_2 - x_3 + 2x_5 & = 7 \\ & x_4 & & - 2x_2 + 4x_3 & = 12 \\ & & x_6 & - 4x_2 + 3x_3 + 8x_5 & = 10 \end{array}$$

Our objective function is

$$c_1 x_1 + c_2 x_2 + c_3 x_3 + c_4 x_4 + c_5 x_5 + c_6 x_6 = x_2 - 8x_3 + 2x_5$$

Hence, $c_1 = 0$; $c_2 = 1$; $c_3 = -3$; $c_4 = 0$; $c_5 = 2$; $c_6 = 0$. We add this information to our tableau by writing c_j above P_j to obtain:

0	0	0	1	-3	2	
P_1	P_4	P_6	P_2	P_3	P_5	P_0
1	0	0	3	-1	2	7
0	1	0	-2	4	0	12
0	0	1	-4	3	8	10

Since P_1, P_4, and P_6 are our basis vectors (which we have indicated by listing them first), we include a column on the left headed "basis" and then list the basis vectors again with the appropriate c_j to the left of basis vector P_j. We then have

		0	0	0	1	-3	2	
T	Basis	P_1	P_4	P_6	P_2	P_3	P_5	P_0
0	P_1	1	0	0	3	-1	2	7
0	P_4	0	1	0	-2	4	0	12
0	P_6	0	0	1	-4	3	8	10

We have designated by vector T, the column of c_j's corresponding to the basis vectors.

We now calculate the real number $(T \cdot P_j) - c_j$ for $j = 0, 1, 2, \ldots, 6$ and list this value beneath the vector P_j. Recall from Section 3.1, Part I, that the dot product of two vectors is a real number. Thus $T \cdot P_j$ is a real number and since c_j is also a real number, $(T \cdot P_j) - c_j$ is a real number. Note that $T \cdot P_j = 0$ for $j = 0, 1, 2, 3, 4, 5, 6$ in this case since $T = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$

Including this final information, we construct "Tableau 1"

TABLEAU 1

		0	0	0	1	-3	2	
T	Basis	P_1	P_4	P_6	P_2	P_3	P_5	P_0
0	P_1	1	0	0	3	-1	2	7
0	P_4	0	1	0	-2	4	0	12
0	P_6	0	0	1	-4	3	8	10
		0	0	0	-1	3	-2	

The array which is heavily outlined will be called the <u>augmented matrix of TABLEAU 1</u>. The circled element will be discussed shortly.

Recall from Section 3.2, Part II, that to find the optimal solution, we merely have to investigate the $\binom{n}{m}$ ways of choosing the m basis vectors from the set of n column vectors of A. If we choose this set of m vectors out of a collection of n vectors in the "best" way, we will have determined our optimal solution.

Thus let $P_s, P_r, \ldots, P_t$ be the m vectors which are the "best" basis vectors. This implies that $x_s, x_r, \ldots, x_t$ are our "best" basis variables. That is, by letting the remaining n-m variables be equal to zero and solving the resulting system of m equations in m unknowns

$$DX = B,$$

we will obtain our optimal solution

$$x = D^{-1}B.$$

The problem, then, is to find the "best" m basis vectors.

With reference to our example, we shall choose a new basis and construct TABLEAU 2 from TABLEAU 1, choose a different basis and construct TABLEAU 3 from TABLEAU 2, etc. However, we shall not determine our new basis vectors in a haphazard manner. Indeed, this is the essence of the simplex method. That is, <u>for each choice of basis vectors, we construct a tableau</u>.

In the simplex method, then, we:

(1) Start with a given basis and construct TABLEAU 1.

(2) Examine each tableau and from the last row, determine

if the basis vectors which gave rise to this tableau are the "best" choice or not.

(3) If not, determine which vectors to designate as the basis vectors for the next tableau. The method here is that the new set of basis vectors will differ from the old set of basis vectors by only one vector. By examining the tableau, we decide which vector is to leave the old basis and which vector is to enter the new basis to replace it.

We illustrate the procedure with our example before formally listing the technique. Recall that we had previously decided that our first set of basis vectors would be P_1, P_4, and P_6 from which we constructed TABLEAU 1.

We examine the last row and see that it is not true that $(T \cdot P_j) - c_j \leq 0$ for all j. That is, $(T \cdot P_3) - c_3 > 0$. The basis corresponding to this tableau, is not the "best" choice of basis vectors.

We now look at $T \cdot P_j - c_j$ for those j for which $(T \cdot P_j) - c_j > 0$ and choose the maximum $(T \cdot P_j) - c_j$. In this case, $(T \cdot P_j) - c_j > 0$ only for j = 3. Hence, $\max (T \cdot P_j) - c_j = (T \cdot P_3) - c_3 = 3$. We therefore select vector P_3 to enter our basis.

We now consider the components of P_3 in this tableau which are positive. We divide these components into the corresponding components of P_0 in this tableau. That is, the components of P_3 which are positive are the second, 4, and the third, 3.

We calculate:

$$\frac{\text{Second component of } P_0}{\text{Second component of } P_3} = \frac{12}{4} = 3$$

$$\frac{\text{Third component of } P_0}{\text{Third component of } P_3} = \frac{10}{3}$$

Since min (3, 10/3) = 3, we circle the second component of P_3. This element, 4, in our tableau is called our pivot element. Looking to the left, we see that the vector P_4 is to leave our basis. Looking above, we see that the vector P_3 is to enter the basis.

Our new set of basis vectors, then, will be P_1, P_3, and P_6. This will determine TABLEAU 2.

We begin construction of TABLEAU 2 by listing the new basis vectors first in our horizontal list across the top as well as listing them in a column to the left along with the appropriate c_j. We obtain:

TABLEAU 2 (partial)

		0	-3	0	1	0	2	
T	Basis	P_1	P_3	P_6	P_2	P_4	P_5	P_0
0	P_1	1	-1	0	3	0	2	7
-3	P_3	0	4	0	-2	1	0	12
0	P_6	0	3	1	-4	0	8	10

Before constructing TABLEAU 2, we wish to express each of the vectors P_0, P_1, P_2, P_3, P_4, P_5, P_6 in terms of the basis vectors P_1, P_3, P_6. From Section 3.2, Part I, we know that this amounts to merely putting the matrix

$$\begin{pmatrix} 1 & -1 & 0 & 3 & 0 & 2 & 7 \\ 0 & 4 & 0 & -2 & 1 & 0 & 12 \\ 0 & 3 & 1 & -4 & 0 & 8 & 10 \end{pmatrix}$$

in its row reduced echelon form to obtain:

$$\begin{pmatrix} 1 & 0 & 0 & 5/2 & 1/4 & 2 & 10 \\ 0 & 1 & 0 & -1/2 & 1/4 & 0 & 3 \\ 0 & 0 & 1 & -5/2 & -3/4 & 8 & 1 \end{pmatrix}$$

We call this last matrix the augmented matrix of TABLEAU 2.

Calculating $T \cdot P_j - c_j$ for $j = 1, 3, 6, 2, 4, 5$ (where $T = \begin{pmatrix} 0 \\ -3 \\ 0 \end{pmatrix}$ in this case), we obtain:

$$\begin{aligned}
T \cdot P_1 &= (0)(1) + (-3)(0) + (0)(0) &&= 0 \\
T \cdot P_3 &= (0)(0) + (-3)(1) + (0)(0) &&= -3 \\
T \cdot P_6 &= (0)(0) + (-3)(0) + (0)(1) &&= 0 \\
T \cdot P_2 &= (0)(5/2) + (-3)(-1/2) + (0)(-5/2) &&= 3/2 \\
T \cdot P_4 &= (0)(1/4) + (-3)(1/4) + (0)(-3/4) &&= -3/4 \\
T \cdot P_5 &= (0)(2) + (-3)(0) + (0)(8) &&= 0
\end{aligned}$$

Hence,

$$\begin{aligned}
T \cdot P_1 - c_1 &= 0 - 0 = 0 \\
T \cdot P_3 - c_3 &= -3 - (-3) = 0 \\
T \cdot P_6 - c_6 &= 0 - 0 = 0 \\
T \cdot P_2 - c_2 &= 3/2 - 1 = 1/2 \\
T \cdot P_4 - c_4 &= -3/4 - 0 = -3/4 \\
T \cdot P_5 - c_5 &= 0 - 2 = -2
\end{aligned}$$

We therefore have:

TABLEAU 2

T	Basis	P_1	P_3	P_6	P_2	P_4	P_5	P_0
		0	-3	0	1	0	2	
0	P_1	1	0	0	5/2	1/4	2	10
-3	P_3	0	1	0	-1/2	1/4	0	3
0	P_6	0	0	1	-5/2	-3/4	8	1
		0	0	0	1/2	-3/4	-2	

Examining the last row, we see that not all of the entries are less than or equal to zero. Thus, our basis vectors are not the "best." Since 1/2 is the only entry greater than zero, it is the largest such entry. Hence P_2 will enter the new basis.

Since 5/2 is the only positive component of P_2, we needn't calculate 10/(5/2) since there is nothing to compare it with. That is, we know immediately that 5/2 will be our pivot element.

Therefore, P_1 will leave our old basis, and P_2 will enter our new basis.

As we did above, before constructing TABLEAU 3, we first rearrange TABLEAU 2 to indicate which vectors will be in our new basis. We obtain:

TABLEAU 3 (partial)

T	Basis	P_2	P_3	P_6	P_1	P_4	P_5	P_0
		1	-3	0	0	0	2	
1	P_2	5/2	0	0	1	1/4	2	10
-3	P_3	-1/2	1	0	0	1/4	0	3
0	P_6	-5/2	0	1	0	-3/4	8	1

Putting the matrix

$$\begin{pmatrix} 5/2 & 0 & 0 & 1 & 1/4 & 2 & 10 \\ -1/2 & 1 & 0 & 0 & 1/4 & 0 & 3 \\ -5/2 & 0 & 1 & 0 & -3/4 & 8 & 1 \end{pmatrix}$$

in its row reduced echelon form, we obtain

$$\begin{pmatrix} 1 & 0 & 0 & 2/5 & 1/10 & 4/5 & 4 \\ 0 & 1 & 0 & 1/5 & 3/10 & 2/5 & 5 \\ 0 & 0 & 1 & 1 & -1/2 & 10 & 11 \end{pmatrix}$$

the augmented matrix of TABLEAU 3.

Calculating $T \cdot P_j - c_j$ for $j = 2, 3, 6, 1, 4, 5$ (where $T = \begin{pmatrix} 1 \\ -3 \\ 0 \end{pmatrix}$ in this case), we obtain:

TABLEAU 3

		1	-3	0	0	0	2	
T	Basis	P_2	P_3	P_6	P_1	P_4	P_5	P_0
1	P_2	1	0	0	2/5	1/10	4/5	4
-3	P_3	0	1	0	1/5	3/10	2/5	5
0	P_6	0	0	1	1	-1/2	10	11
		0	0	0	-1/5	-4/5	-12/5	

Examining the last row, we see that all values of $T \cdot P_j - c_j$ are less than or equal to zero. The basis vectors which gave rise to this tableau, then, are the "best."

Now TABLEAU 3 corresponds to the following system which is equivalent to our original system because of the derivation of this tableau:

$$
\begin{aligned}
x_2 \qquad\qquad &+ 2/5\ x_1 + 1/10\ x_4 + 4/5\ x_5 = 4 \\
x_3 \qquad &+ 1/5\ x_1 + 3/10\ x_4 + 2/5\ x_5 = 5 \\
x_6 &+ \quad x_1 - 1/2\ x_4 + 10\ x_5 = 11
\end{aligned}
$$

or rewriting this:

$$
\begin{aligned}
2/5\ x_1 + x_2 \qquad &+ 1/10\ x_4 + 4/5\ x_5 \qquad = 4 \\
1/5\ x_1 \qquad + x_3 &+ 3/10\ x_4 + 2/5\ x_5 \qquad = 5 \\
x_1 \qquad\qquad &- 1/2\ x_4 + 10\ x_5 + x_6 = 11
\end{aligned}
$$

Since P_2, P_3, and P_6 were the vectors which we decided were the best basis vectors, we choose x_2, x_3, and x_6 as the basis variables. That is, we set $x_1 = x_4 = x_5 = 0$. This reduces our system to:

$$
\begin{aligned}
x_2 \qquad\quad & 4 \\
x_3 \qquad & 5 \\
x_6 &= 11
\end{aligned}
$$

which we trivially solve to obtain:

$x_2 = 4;\ x_3 = 5;\ x_6 = 11.$

Hence our optimal solution is

$$
X = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{pmatrix} = \begin{pmatrix} 0 \\ 4 \\ 5 \\ 0 \\ 0 \\ 11 \end{pmatrix}
$$

We can easily calculate the optimal value of the objective function

$$CX = (c_1, c_2, \ldots, c_6) \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_6 \end{pmatrix} = (0, 1, -3, 0, 2, 0) \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{pmatrix}$$

by substitution to obtain:

$$(0, 1, -3, 0, 2, 0) \begin{pmatrix} 0 \\ 4 \\ 5 \\ 0 \\ 0 \\ 11 \end{pmatrix}$$

$$= (0)(0) + (1)(4) + (-3)(5) + (0)(0) + (2)(0) + (0)(11)$$

$$= -11$$

Let us now state the procedure which we have followed.

Minimize $CX = (c_1, c_2, \ldots, c_n) \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = c_1 x_1 + c_2 x_2 + \ldots + c_n x_n$ subject to the restraints:

$$\text{(1)} \quad \begin{aligned} a_{11} x_1 + a_{12} x_2 + \ldots + a_{1n} x_n &= b_1 \\ a_{21} x_1 + a_{22} x_2 + \ldots + a_{2n} x_n &= b_2 \\ &\vdots \\ a_{m1} x_1 + a_{m2} x_2 + \ldots + a_{mn} x_n &= b_m \end{aligned}$$

$$\text{(2)} \quad x_i \geq 0; \; i = 1, 2, \ldots, n$$

We consider the vectors $P_1, P_2, \ldots, P_n$ and attempt to find the "best" selection of m of these vectors to be our basis vectors. We then solve for the corresponding basis variables by letting the remaining n-m variables be equal to zero.

We have:

$$P_1 = \begin{pmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{pmatrix}; \quad P_2 = \begin{pmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{m2} \end{pmatrix}; \; . \; . \; . \; . \; . \; P_n = \begin{pmatrix} a_{1n} \\ a_{2n} \\ \vdots \\ a_{mn} \end{pmatrix}; \quad P_0 = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{pmatrix}$$

Now, for each tableau, our values of a_{ij} and b_j ($i = 1, 2, . . . , n$; $j = 1, 2, . . . , m$) will change. However, this will not cause us any confusion. If we write a_{23} when speaking of the 4-th tableau, we shall mean the second component of P_3 in this tableau. This will differ from a_{23} in the 7-th tableau. Likewise, b_5 in our second tableau will differ from b_5 in our 8-th tableau since the column of constants in the equivalent systems which these tableaus respectively represent may differ.

(1) We begin with a given set of m basis vectors. We then construct a matrix with $P_1, P_2, . . . , P_n, P_0$ as its columns -- putting the m basis vectors first, then the remaining n-m nonbasis vectors. We then put this matrix in its row reduced echelon form if it is not already in this form. This is the augmented matrix for this tableau.

Above each column of this matrix, we designate it by P_j ($j = 0, 1, . . . , n$). Above each P_j ($j = 1, 2, ... , n$), we put the corresponding c_j.

To the left of the 1-st column, we form a column designating the m basis vectors for this tableau with the corresponding c_j forming a column to the left of this. We call this final column the vector T for this tableau.

For $j = 1, 2, \ldots, n$, we calculate the real number $T \cdot P_j - c_j$ and place it in a box beneath the column headed P_j.

(2) We examine the last row of our tableau. If $T \cdot P_j - c_j \leq 0$ for all j, we have arrived at the "best" selection of m basis vectors.

If there exists a j for which $T \cdot P_j - c_j > 0$ but for which $a_{ij} \leq 0$ for all i, then no optimal solution exists.

If for every j for which $T \cdot P_j - c_j > 0$, there exists some $a_{ij} > 0$, then the P_j for which $T \cdot P_j - c_j$ is the largest will be the vector to enter our new basis.

(3) If P_j is the vector determined in step (2) to enter the basis, we know that at least one $a_{ij} > 0$.

For all such i (i.e., for all $a_{ij} > 0$), we calculate

$$\frac{b_i}{a_{ij}}$$

Note that j is fixed. It is determined by (2). The a_{ij} for which the $\frac{b_1}{a_{ij}}$ just calculated is least is called the pivot element of our tableau.

If a_{ij} is the pivot element, P_i is to leave the basis and P_j is to enter the basis.

We now have a new basis to investigate and hence go back to step (1). We continue this process until it is determined in step (2) that we have arrived at our "best" set of m basis vectors or that no optimal solution exists.

If the LP problem was to maximize CX subject to:

(1) $AX = B$

(2) $X \geq 0$,

steps (1) and (3) in our procedure would remain the same. Step (2), however, would become:

(2') We examine the last row of our tableau.

If $T \cdot P_j - c_j \geq 0$ for all j, we have arrived at the "best" set of basis vectors.

If there exists a j for which $T \cdot P_j - c_j < 0$ but for which $a_{ij} \leq 0$ for all i, then no optimal solution exists.

If for every j for which $T \cdot P_j - c_j < 0$, there exists some $a_{ij} > 0$, then the P_j for which $T \cdot P_j - c_j$ is the most negative will be the vector to enter the basis.

Note: The general LP problem is to minimize

$$CX = c_1 x_1 + c_2 x_2 + \dots + c_n x_n$$

subject to the restraints:

$$(1) \quad \begin{array}{l} a_{11} x_1 + a_{12} x_2 + \dots + a_{1n} x_n = b_1 \\ a_{21} x_1 + a_{22} x_2 + \dots + a_{2n} x_n = b_2 \\ \vdots \\ a_{m1} x_1 + a_{m2} x_2 + \dots + a_{mn} x_n = b_m \end{array}$$

$$(2) \quad x_i \geq 0; \quad i = 1, 2, \dots, n$$

We must begin the simple procedure with a given basis. It will often occur, as in the example of this section, that m of the n vectors $P_1, P_2, \dots, P_n$ are

$$\begin{pmatrix}1\\0\\0\\\vdots\\0\end{pmatrix};\ \begin{pmatrix}0\\1\\0\\0\\\vdots\\0\end{pmatrix};\ \cdots\cdots;\ \begin{pmatrix}0\\0\\\vdots\\0\\1\\0\end{pmatrix};\ \begin{pmatrix}0\\0\\\vdots\\0\\1\end{pmatrix}.$$

We would then choose these m vectors as our initial basis vectors. If we do not have m such vectors among our original n, we might consider the following <u>augmented LP problem</u>:

Minimize $c_1 x_1 + c_2 x_2 + \ldots + c_n x_n + wx_{n+1} + wx_{n+2} + \ldots + wx_{n+m}$

subject to:

$$\begin{aligned}
(1)\quad & a_{11} x_1 + a_{12} x_2 + \ldots + a_{1n} x_n + x_{n=1} && = b_1\\
& a_{21} x_1 + a_{22} x_2 + \ldots + a_{2n} x_n + \ldots x_{n=2} + \ldots && = b_2\\
& \vdots \\
& a_{m1} x_1 + a_{m2} x_2 + \ldots + a_{mn} x_n + \ldots\ldots + x_{n+m} && = b_m
\end{aligned}$$

$$(2)\quad x_i \geq 0\ ; \quad i = 1, 2, \ldots n, n+1, n+2, \ldots n+m$$

w is an unspecified large number.

Thus, the vectors $P_{n+1}, P_{n+2}, \ldots, P_{n+m}$ can be chosen as the initial basis for this augmented LP problem. If $X = \begin{pmatrix}x_1\\x_2\\\vdots\\x_n\end{pmatrix}$ is a feasible solution to the original problem, then $X' = \begin{pmatrix}x_1\\\vdots\\x_n\\0\\\vdots\\0\end{pmatrix}$ is a feasible solution to the augmented problem.

By the nature of w, if the simplex procedure gives us a

minimum solution $X' = \begin{pmatrix} x_1 \\ x_2 \\ \cdot \\ x_n \\ \vdots \\ x_{n+m} \end{pmatrix}$ to the augmented system, then

$$x_{n+1} = x_{n+2} = \ldots = x_{n+m} = 0.$$

$X = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}$ would then be the optimal solution to our original problem.

If the original problem is not feasible, and if $X' = \begin{pmatrix} x_1 \\ \vdots \\ x_n \\ x_{n+1} \\ \vdots \\ x_{n+m} \end{pmatrix}$ is the minimum feasible solution to the augmented system, then at least one x_{n+i} will be positive. The interested reader is referred to Gass, page 72.

In models which arise from LP problems in which a slack variable has to be introduced in each equation (i.e., when we have a system of inequalities), there will be no need to worry about an augmented LP problem. The basis vectors will be apparent.

The reader, with a great deal of time and patience, should <u>now</u> be able to find the optimal solutions to the examples in Chapter 2, Part II. It will prove to be more efficient, however, to utilize an electronic computer to obtain these solutions. Part III discusses the necessary procedure.

PART III: UTILIZATION OF THE UNIVAC 1108

CHAPTER 1: OBTAINING AN OPTIMAL SOLUTION (THE PRIMAL PROBLEM)

1.1 Scope

This manual is a primer to be used in conjunction with the Linear Programming--Programmer's Reference Manual (UP-4138). The scope of this manual is especially pertinent in Part II. No attempt is made to discuss all of the commands mentioned in UP-4138. Instead, a few basic commands are explained with the hope that the user will be able to utilize manual UP-4138 after the introduction presented here.

An LP tape is available for the UNIVAC 1108 located at Bellcomm in L'Enfant Plaza, Washington, D.C. This tape, LPPWB--#1815, contains a program for "solving" an LP problem. To utilize this tape, the user must be able to "call" the tape via the Exec 8 control language, "feed in" the data for his particular problem, and finally obtain the desired information by using various "commands."

In this chapter, we seek only to obtain the optimal solution to our LP problem and to display our data in a convenient form. Finding the optimal solution to an LP problem is sometimes referred to as the primal problem.

We shall investigate other information about our LP problem which can be obtained from our LP tape as well as the commands necessary to obtain this additional information in the next chapter.

1.2 The Run Stream

By the run stream of our LP problem, we shall mean the sequence of commands to the Executive System necessary to utilize the LP tape. The following run stream is appropriate for the Exec 8 System Version 8.2.

The first card in our run stream is our "run card." The information on this card, which may differ for various runs, is explained below.

```
@ RUN,1R    RCTPB1,NHQRCT,LPPWB,15,50
```

Run Identification | Account Number | Project Number | Max Time | Max Pages

Following our run card, we place the remaining Executive commands. They are:

```
@FREE TPF$
```

```
@ASG,T TPF$,F11150
```

```
@ASG,T LPTAPE,T,1815

@ASG,T 9,T,SCRTCH

@FIND,A LPTAPE.LP1108/E8S

@COPIN,A LPTAPE.LP1108/E8S,TPF$.

@FREE LPTAPE

@XQT .LP1108/E8S
```

The reader is refered to the Bellcomm--Exec 8--User's Manual for an explanation of these commands.

Following the last Executive command above, the "Execute" command (@XQT - etc.), come the commands to the 1108 and the data. After all of these cards are arranged in the deck, place a final Executive command. This card is simply

1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26	27	28	29	30	31	32	33	34	35	36	37
@	F	I	N																																	

It follows the card

1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26	27	28	29	30	31	32	33	34	35	36	37	38	39	40	41
E	N	D	J	O	B																																			

1.3 Input of Data

Recall Example 1, Chapter 2, Part II. We wished to determine x_1, x_2, and x_3, the number of units of products 1, 2, and 3, respectively, which should be produced each week in order to maximize the profit function

$$20x_1 + 6x_2 + 8x_3$$

subject to the restrictions:

$$\begin{array}{lrcrcrcl}
(1) & 8x_1 & + & 2x_2 & + & 3x_3 & \leq & 200 \\
 & 4x_1 & + & 3x_2 & & & \leq & 100 \\
 & 2x_1 & & & + & x_3 & \leq & 50 \\
 & & & & & x_3 & \leq & 20
\end{array}$$

$$(2) \quad x_i \geq 0; \quad i = 1, 2, 3$$

Note here that we leave condition (1) in the form of a system of inequalities.

We first create a name for each row and for each variable in (1) as well as a name for the objective function. The requirement here is that each name must contain from one to six symbols with a space counting as a symbol.

For the three variables, x_1, x_2, and x_3, we create names which will remind us what these variables represent. Hence, we let:

PROD 1 = x_1

PROD 2 = x_2

PROD 3 = x_3

The obvious name for our objective function is the six symbol name: PROFIT.

Considering the origin of the four inequalities in (1), we designate them as follows:

$$\begin{array}{llll} 8x_1 + 2x_2 + 3x_3 \leq 200 & & & \text{MILMCH} \\ 4x_1 + 3x_2 \qquad\quad \leq 100 & & & \text{LATHE} \\ 2x_1 \qquad\quad + x_3 \leq 50 & & & \text{GRNDER} \\ \qquad\qquad\quad x_3 \leq 20 & & & \text{SALPT3} \end{array}$$

Recall that after the introduction of slack variables, our system of inequalities became the following system of equations:

$$\begin{array}{rcl} 8x_1 + 2x_2 + 3x_3 + x_4 & = & 200 \\ 4x_1 + 3x_2 \qquad\qquad + x_5 & = & 100 \\ 2x_1 \qquad + x_3 \qquad\qquad + x_6 & = & 50 \\ x_3 \qquad\qquad\qquad + x_7 & = & 20 \end{array}$$

We will not be concerned with writing this system of equations when utilizing the computer. The computer will "take care of this" for us. We need not create a name for the slack variables x_4, x_5, x_6, and x_7. The computer will designate them by the name of the inequality from which they arose. That is, the computer will set:

$$\begin{array}{l} \text{MILMCH} = x_4 \\ \text{LATHE} = x_5 \\ \text{GRNDER} = x_6 \\ \text{SALPT3} = x_7 \end{array}$$

* * Therefore, MILMCH represents both the first inequality and the slack variable introduced into this inequality to create an equation. * *

Hence, if MILMCH assumes a positive value in our optimal solution, this means that the slack variable x_6 is positive (rather than zero) in our optimal solution. We noted in Section 2.1, Part II, that this, in turn, would imply x_6 hours per week out of the 200 hours available on the milling machine should not be used.

We are now prepared to input the data for this particular problem. The first card after the executive command "execute", is a control command to the 1108 designating whether the objective function is to be maximized or minimized.

* If this card is absent, the system will assume that the objective function is to be maximized. * *

Since this is indeed the case in our problem, we do not include this card.

If the objective function were to be minimized, we would have included the following card after the "@XQT - etc." card:

1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26	27	28	29	30	31	32	33	34	35	36
S	E	T		O	B	J	E	C	T		T	O		(	M	I	N				)														

Suppose that our objective function is $c_1 x_1 + c_2 x_2 + \ldots + c_n x_n$ and our (1st) restriction is

$$
\begin{aligned}
&a_{11} x_1 + a_{12} x_2 + \ldots + a_{1n} x_n \leq b_1\\
&a_{21} x_1 + a_{22} x_2 + \ldots + a_{2n} x_n \leq b_2\\
&\vdots\\
&a_{m1} x_1 + a_{m2} x_2 + \ldots + a_{mn} x_n \leq b_m
\end{aligned}
$$

We have indicated that the inequalities may be in either direction. We may also have equalities.

To read this information into the computer, we must specify the values of $c_1, a_{11}, a_{21}, \ldots, a_{m1}, c_2, a_{12}, a_{22}, \ldots, a_{m2}, c_3, \ldots\ldots, c_n, a_{1n}, a_{2n}, \ldots, a_{mn}, b_1, b_2, \ldots, b_m$ <u>in this order</u>.

We must have one card for each of these values and must arrange these cards in the indicated order. A card need <u>not</u> be included in the data deck, however, to specify a value of <u>zero</u>.

We illustrate this procedure with an example. Following the card "SET OBJECT TO (MIN)", if it is included, we put the card

1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26	27	28	29	30	31	32	33	34	35	36	37	38	39	40
L	O	A	D																																				

Our next card(s) designates the names we have created for our rows (i.e.,our objective function and the inequalities in condition (1)). We have:

1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26	27	28	29	30	31	32	33	34	35	36	37	38
R	O	W		I	D																																

one card

```
     5      PROFIT     +MILMCH     +LATHE      +GRNDER     +SALPT3
```

Note the card columns in which we enter the appropriate information. The "5" in column 6 represents the number of row labels contained on the card. We begin the row names in columns 25, 37, 49, and 61. If there were any more row labels, we would have begun another card putting the appropriate number in column 6 and again starting the labels in columns in 25, 37, etc.

Before each row label, we put either a "+", a "-", or a blank:

A blank indicates that our row is either an <u>equation</u> or the objective function.

A "+" indicates that our row is an inequality of the form "$\leq$".

A "-" indicates that our row is an inequality of the form "$\geq$".

We proceed to input the <u>nonzero</u> c_j and a_{ij} as follows.

We include the card

```
MATRIX
```

before specifying our data.

Our first data card is

```
      PROD1PROFIT    20.
```

which represents the coefficient of x_1 (PROD 1) in the objective function (PROFIT).

Pay particular attention to the card columns in which this information is placed and especially to the location of the decimal point. Our next few cards are:

1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26	27	28	29	30	31	32	33	34	35	36	37	38	39	40	41
						P	R	O	D		1	M	I	L	M	C	H					8	.																	

1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26	27	28	29	30	31	32	33	34	35	36	37	38	39	40	41
						P	R	O	D		1	L	A	T	H	E						4	.																	

1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26	27	28	29	30	31	32	33	34	35	36	37	38	39	40	41
						P	R	O	D		1	G	R	N	D	E	R					2	.																	

Here we have indicated the coefficient of x (PROD 1) in the third inequality (GRNDER). We do not include a card for the coefficient of x in the fourth inequality (SALPT3) since this coefficient is zero.

Our remaining cards are then:

1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26	27	28	29	30	31	32	33	34	35	36	37	38
						P	R	O	D		2	P	R	O	F	I	T					6	.														

1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26	27	28	29	30	31	32	33	34	35	36	37	38
						P	R	O	D		2	M	I	L	M	C	H					2	.														

```
      PROD 2LATHE     3.
```

```
      PROD 3PROFIT    8.
```

```
      PROD 3MILMCH    3.
```

```
      PROD 3GRNDER    1.
```

```
      PROD 3SALPT3    1.
```

We now input the column of constants (called the B-vector) by first including the card

```
FIRSTB
```

and the data cards

```
            MILMCH 200.
```

```
            LATHE   100.
```

```
            GRNDER   50.
```

```
            SALPT3   20.
```

Note that we indicate which row the constant on any card corresponds to by merely listing the row label in the appropriate card columns. Our next card is

```
ENDATA
```

and indicates that we have completed our input.

1.4 Display Commands

After we have input our data, and before we ask for the optimal solution (our primal solution), we may wish to analyze and display the data. We may include the following commands in the deck of cards.

```
MATRIX
```

1	2	3	4	5	6
E	Q	L	I	S	T

1	2	3	4	5	6
M	T	X	M	A	P

The commands MATRIX and EQLIST merely cause our data to be displayed in matrix form and equation form respectively. "Matrix" is discussed in Section 5, page 66 and in Appendix A, page 11 of UP-4138. "EQLIST" is discussed in Section 5, page 28 and in Appendix A, page 13 of UP-4138.

The command MTXMAP causes the matrix displayed by the command MATRIX to be analyzed and printed in a coded pictorial form. This command is discussed in Section 5, page 67 and in Appendix A, page 15 of UP-4138.

The output effected by these commands is given on the next four pages, the third and fourth pages of output corresponding to the command MTXMAP.

With reference to the matrix tableau, the names we have given to the objective function and the restraints of condition (1) are listed in the column at the left. The names we have given to the variables x_1, x_2, and x_3 are listed in the top row.

The element in the row labeled LATHE and the column labeled PROD 2 is 3.000000. This corresponds to the fact that "3" is the coefficient of x_2 (represented by PROD 2) in the second restraint of condition (1) (represented by LATHE).

CASE MATRIX TABLEAU

LABEL	COST	B-VEC	PROD 1	PROD 2	PROD 3	*B1
COST	.000000	.[illegible]00000	20.000000	6.000000	4.000000	.000000
ROW LABEL						
1 PROFIT	.000000	.100000	20.000000	6.000000	8.000000	.000000
2 MILMCH	.000000	200.000000	8.000000	2.000000	3.000000	200.000000
3 LATHE	.000000	100.000000	4.000000	3.000000	.000000	100.000000
4 GRNDER	.000000	50.000000	2.000000	.000000	1.000000	50.000000
5 SALPT3	.000000	20.000000	.000000	.000000	1.000000	20.000000

END OF MATRIX TABLEAU

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```
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CASE                          EQLIST OUTPUT

EQUATION    1 LABEL PROFIT COST =     .000000 ORIGINAL BI =     .000000
 +  1.00000(PROFIT) +  20.00000(PROD 1) +   6.00000(PROD 2) +   8.00000(PROD 3)

EQUATION    2 LABEL MILNCH COST =     .000000 ORIGINAL BI =  200.000000
 +  1.00000(MILNCH) +   4.00000(PROD 1) +   2.00000(PROD 2) +   3.00000(PROD 3) + 200.00000(*B1    )

EQUATION    3 LABEL LATHE  COST =     .000000 ORIGINAL BI =  100.000000
 +  1.00000(LATHE ) +   4.00000(PROD 1) +   3.00000(PROD 2) + 100.00000(*B1    )

EQUATION    4 LABEL GRNDER COST =     .000000 ORIGINAL BI =   50.000000
 +  1.00000(GRNDER) +   2.00000(PROD 1) +   1.00000(PROD 3) +  50.00000(*B1    )

EQUATION    5 LABEL SALPT3 COST =     .000000 ORIGINAL BI =   20.000000
 +  1.00000(SALPT3) +   1.00000(PROD 3) +  20.00000(*B1    )

END EQLIST OUTPUT
```

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```
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              C B P P P .
              U - R R R B
              S V 0 0 0 1
              T E 0 0 0
                C
                  1 2 3

  COST        . . 7 6 6 .
PROFIT        . . 7 6 6 .
MILMCH        . 6 6 6 6 8
LATHE         . 7 6 6 . 7
GRINDER       . 7 6 . 1 7
SALPT3        . 7 . . 1 7
```

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THE MATRIX ELEMENTS ARE REPRESENTED AS FOLLOWS

CODE	MAGNITUDE OF ELEMENTS GREATER THAN	EQUAL TO OR LESS THAN	OCCURENCES
0	0.0	0.0001	0
2	0.0001	0.001	0
3	0.001	0.01	0
4	0.01	0.1	0
5	0.1	0.99999999	0
1	=1.0		2
6	1.0	10.0	10
7	10.0	100.0	8
8	100.0	1000.0	2
9	1000.0		0

END MTXMAP OUTPUT

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Note that the column headed B-VEC (and also the column headed *B1) contains the right-hand side elements of the restraints of condition (1).

This matrix corresponds to an LP problem in which the objective function is to be maximized and the restraints are all inequalities of the "$\leq$" order. Compare this matrix with the matrix in Section 3.2, Part III. In that case, the matrix corresponds to an LP problem in which the objective function is to be minimized and in which the restraints of condition (1) are inequalities of the "$\geq$" order.

For illustrative purposes, note that to find x_1, x_2, and x_3 which maximize

$$x_1 - 2x_2 + 3x_3$$

is equivalent to finding x_1, x_2, and x_3 which minimize

$$-(x_1 - 2x_2 + 3x_3) = -x_1 + 2x_2 - 3x_3.$$

Also, x_1, x_2, and x_3 are such that

$$4x_1 + x_2 - 5x_3 \geq 10$$

if and only if

$$-4x_1 - x_2 + 5x_3 \leq -10$$

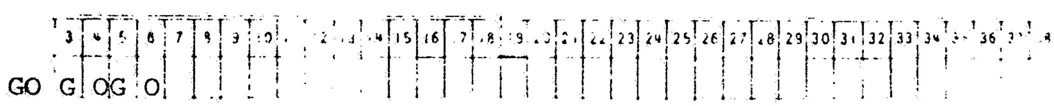
From the MTXMAP output, we read that the element in the row labeled MILMCH and the column labeled PROD 2 of the MATRIX OUTPUT for this problem should be greater than or equal to 1.0 but less than or equal to 10. This is indeed the case since a glance at our matrix output reveals that this element is 2.0.

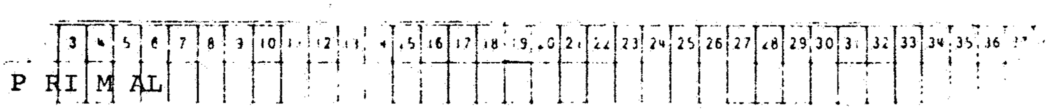
Also, the MTXMAP output tells us that the element in the row labeled LATHE and the column labeled PROD 3 of our MATRIX output should be a zero. A glance at our MATRIX output confirms this.

1.5 GOGOGO

The command GOGOGO causes an optimal solution to be obtained if such a solution exists. The activity of the 1108 which is triggered by this command replaces the calculations discussed in Part II. It is therefore an extremely important command. For a further discussion, see Section 5, page 39 of UP-4138.

We follow the GOGOGO command with the command PRIMAL. This causes the optimal solution arrived at by the command to be printed. The two cards which follow our display commands are therefore:





Recall our example of maximizing the profit function $20x_1 + 6x_2 + 8x_3$ subject to:

$$
\begin{array}{lrcrcrcl}
(1) & 8x_1 & + & 2x_2 & + & 3x_3 & \leq & 200 \\
 & 4x_1 & + & 3x_2 & & & \geq & 100 \\
 & 2x_1 & & & + & x_3 & \leq & 50 \\
 & & & & & x_3 & \leq & 20
\end{array}
$$

(2) $x_i \geq 0$; $i = 1, 2, 3$

The optimal solution to this problem is read from the output on the following page.

Interpreting these results, we see that, in order to maximize profit, the manufacturing firm (Section 2.1, Part II) should produce (on the average) 13 3/4 units of product 1 per week, 15 units of product 2 per week, and 20 units of product 3 per week.

Note that GRNDER = x_6 takes on the nonzero value 2.5 in the optimal solution. Recall that we introduced x_6 into the inequality

to obtain the $\quad 2x_1 + x_3 \leq 50$

equation $\quad 2x_1 + x_3 + x_6 = 50$

The "best" value of x_6 is 2.5. This means simply that the manufacturer should not use 2.5 machine hours per week of the 50 machine hours per week available on the grinder.

We therefore have that x_1, x_2, x_3, and x_6 are our basis variables. We can easily calculate the optimal value of our objective function (the maximum weekly profit) by calculating

$$c_1 x_1 + c_2 x_2 + c_3 x_3 + c_4 x_4 + c_5 x_5 + c_6 x_6$$
$$= 20(13\ 3/4) + 6(15) + 8(20) + 0(0) + 0(2.5) + 0(0)$$
$$= \$525$$

However, this information is contained in our primal output. We merely look for the objective function label and read the negative of the "Activity Number" beside it.

20 AUG 70 .001.013.001

PRIMAL OUTPUT

CASE ITERATION 4 OBJECTIVE VALUE 525.00000

	LABEL	COST	ACTIVITY	LABEL	COST	ACTIVITY	LABEL	COST	ACTIVITY
E	PROFIT	.000000	-.5250000+03	GRNDER	.000000	2.500000	PROD 1	20.000000	13.750000
	PROD 2	6.000000	15.000000	PROD 3	8.000000	20.000000			

END PRIMAL OUTPUT

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In our case we note that our objective function label, PROFIT, has an "E" to the left of it. We therefore read -.5250000+03 by moving the decimal point 3 places to the right to obtain -525.0000. The optimal value of our objective function (maximum weekly profit) is therefore $525.

Note also that the "cost" is listed next to the name created for each variable. The "cost" listed next to the name created for variable x_j is nothing more than c_j, the coefficient of x_j in the objective function.

CHAPTER 2: POST OPTIMAL ANALYSIS

2.1 The Dual Problem

Aside from the fact that the utilization of an electronic computer to obtain the optimal solution to an LP problem is much more efficient than performing the simplex method by hand, additional information can be obtained from the 1108 after the optimal solution has been found. For the most part, this additional information is of the "what if . . . " nature. That is, suppose we alter our original problem in some manner. What can we then say about our previous results?

We begin this post optimal analysis by considering the dual problem.

Let our general LP problem be to find x_i; $i = 1, 2, \dots, \underline{n}$ which <u>maximize</u>

$$c_1 x_1 + c_2 x_2 + \dots + c_n x_n$$

subject to:

(1) $$a_{11} x_1 + a_{12} x_2 + \dots + a_{1n} x_n \leq b_1$$
$$a_{21} x_1 + a_{22} x_2 + \dots + a_{2n} x_n \leq b_2$$
$$\vdots$$
$$a_{m1} x_1 + a_{m2} x_2 + \dots + a_{mn} x_n \leq b_m$$

(2) $x_i \geq 0$; $i = 1, 2, \dots, n$

We shall call the above LP problem our Primal Problem. Corresponding to this primal problem, we have the following Dual Problem.

Find y_i; $i = 1, 2, \ldots, m$ which minimize

$$b_1 y_1 + b_2 y_2 + \ldots + b_m y_m$$

subject to:

$$(1) \quad \begin{array}{l} a_{11} y_1 + a_{21} y_2 + a_{31} y_3 + \ldots + a_{m1} y_m \geq c_1 \\ a_{12} y_1 + a_{22} y_2 + a_{32} y_3 + \ldots + a_{m2} y_m \geq c_2 \\ \vdots \\ a_{1n} y_1 + a_{2n} y_2 + a_{3n} y_3 + \ldots + a_{mn} y_m \geq c_n \end{array}$$

$$(2) \quad y_i \geq 0; \quad i = 1, 2, \ldots, m$$

Note that we have changed form, maximizing to minimizing the objective function and that in (1), we have changed the order of inequalities as well as interchanging the rows and columns of coefficients.

EXAMPLE: Recall that the example with which we have been working in Chapter 1, Part III, is:

$$2x_1 + 6x_2 + 8x_3$$

Maximize subject to:

$$(1) \quad \begin{array}{l} 8x_1 + 2x_2 + 3x_3 \leq 200 \\ 4x_1 + 3x_2 \leq 100 \\ 2x_1 + x_3 \leq 50 \\ x_3 \leq 20 \end{array}$$

$$(2) \quad x_i \geq 0; \quad i = 1, 2, 3$$

Our dual problem to this primal problem then is:

Minimize

$$200y_1 + 100y_2 + 50y_3 + 20y_4$$

subject to:

$$\begin{aligned} (1) \quad & 8y_1 + 4y_2 + 2y_3 \geq 2 \\ & 2y_1 + 3y_2 \geq 6 \\ & 3y_1 + y_3 + y_4 \geq 8 \end{aligned}$$

$$(2) \quad y_i \geq 0; \quad i = 1, 2, 3, 4$$

If the general LP problem (page 124) is such that a few of the inequalities of condition (1) differ in order, the situation can be remedied by multiplying by a negative one.

If any of the restraints of (1) are equations rather than inequalities, we adjust the dual problem appropriately. That is, if the i-th restraint ($i = 1, 2, . . . , m$) of condition (1) in the primal problem is an equation, we remove the nonnegativity requirement from the i-th variable, y_i, in the dual problem.

Likewise, if the i-th variable, x_i, of the primal problem is not required to be nonnegative by condition (2), then the i-th restraint in condition (1) of the dual problem is an equation.

This may seem a bit complicated, and indeed, duality theory can become very complex.

* * The important thing to note here is that to every restraint in condition (1) of the primal problem there corresponds a variable in the dual problem. * * *

It seems, therefore, that information about the restraints in (1) of our LP problem (the primal) should be obtained from the solution to the dual problem. This information is obtained by merely including the card

```
DUAL
```

in our deck after the command, PRIMAL.

An interpretation of the significance of the solution to the dual problem can be found in Gass, page 93. With reference to our example, the 1108 will interpret the dual solution and print the output given on the following page.

We read from this that if the right-hand side of the restraint corresponding to MILMCH were increased by one unit, then the objective function would increase by 2.25 units, the listed <u>shadow price</u>. If the right-hand side of this restraint decreased by one unit, then the objective function would decrease by 2.25 units. That is, with reference to our original problem, Section 2.1, Part II, <u>if</u> there were 200 + 1 = 201 machine hours per week available on the milling machine, the optimal solution would yield a value of \$525.00 + \$2.25 = \$527.25 for the profit function. If there were 200 - 1 = 199 machine hours per week available on this machine, the optimal value of the profit function would be \$525.00 - \$.50 = \$524.50.

Also, if the sales potential for product 3 were 20 + 1 = 21 units per week, the optimal value of the objective function would

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DUAL OUTPUT

CASE ITERATION 4 OBJECTIVE VALUE 525.00000

	LABEL	COST	SHADOW PRICE	LABEL	COST	SHADOW PRICE	LABEL	COST	SHADOW PRICE
E	PROFIT	.000000	.000000	MILMCH	.000000	2.250000	LATHE	.000000	.500000
	GRNDER	.000000	.000000	SALPT3	.000000	1.250000			

END DUAL OUTPUT

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be \$525.00 + \$1.25 = \$526.25; while if this sales potential were 20 - 1 = 19 units per week, the optimal value of the profit function would be \$523.75.

A change in the machine hours per week available on the grinder, however, would have no effect on the optimal value of the profit function.

If a negative shadow price had appeared, an increase in the number to the right of the "$\leq$" sign in the corresponding restraint would cause a decrease in the optimal value of the profit function.

It should be noted that we wished to maximize our objective function in this problem. If the objective function were to be minimized in an LP problem, a positive shadow price listed next to a restraint label would imply that an increase in the number on the right-hand side of the "$\leq$" sign for this restraint would cause the optimal value of the objective function to decrease. Likewise, a negative shadow price for a restraint in a minimization problem would imply that an increase in the number to the right of the "$\leq$" sign in this restraint would cause the optimal value of the objective function to increase.

2.2 Reduced Cost

We recall from section 3.2, Part II, that the optimal solution to the LP problem:

Maximize $$c_1 x_1 + c_2 x_2 + \dots + c_n x_n$$

subject to:

$$(1)\quad \begin{aligned} a_{11}\,x_1 + a_{12}\,x_2 + \cdots\cdots + a_{1n}\,x_n &= b_1 \\ a_{21}\,x_1 + a_{22}\,x_2 + \cdots\cdots + a_{2n}\,x_n &= b_2 \\ &\vdots \\ a_{m1}\,x_1 + a_{m2}\,x_2 + \cdots\cdots + a_{mn}\,x_n &= b_m \end{aligned}$$

$$(2)\quad x_i \geq 0; \quad i = 1, 2, \ldots, n$$

is <u>determined</u> by the proper choice of the m basis variables. That is, if we know which m variables are our basis variables for the optimal solution, this solution can be easily found by setting the remaining n-m variables equal to zero and solving the resulting system of m equations in m unknowns (for which there exists a unique solution by the assumption of Section 3.1, Part II).

Assume then, that we have obtained an optimal solution to the above LP problem. Suppose now that this LP problem is altered somewhat. That is, we are faced with an LP problem which is similar to our original except that a change in the physical phenomenon for which it is a model effect a slight change in the coefficients of the objective function (the c_j) or a slight change in the right hand side values of condition (1) (the b_j).

The question arises: Will the basis variables for our optimal solution to our new problem differ from the basis variables for the optimal solution to our original problem?

In particular, the Reduced Cost output, obtained by including the card

	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26	27	28	29	30	31	32	33	34	35	36	37
RE	D	C	S	T																															

in our deck after the DUAL command, answer the following question:

* * Suppose x_j is not a basis variable for the optimal solution to our LP problem. By what amount must c_j, the coefficient of x_j in the objective function, be reduced before x_j can become a basis variable for the optimal solution to this altered LP problem? * *

The reduced cost output for our example is given on the following page. We read that if our objective function

$$20x_1 + 6x_2 + 8x_3 + 0x_4 + 0x_5 + 0x_6 + 0x_7$$

were changed to

$$20x_1 + 6x_2 + 8x_3 - 2.25x_4 + 0x_5 + 0x_6 + 0x_7,$$

the nonbasis variable, x_4, (represented by MILMCH), of our original LP problem would become a basis variable for our new LP problem.

Likewise, if our objective function were changed to

$$20x_1 + 6x_2 + 8x_3 + 0x_4 - .5x_5 - 0x_6 + 0x_7,$$

the nonbasis variable x_5 (represented by LATHE) would become a basis variable for the optimal solution of our new LP problem.

Also, if our objective function became

$$20x_1 + 6x_2 + 8x_3 + 0x_4 + 0x_5 + 0x_6 - 1.25x_7,$$

the nonbasis variable, x_7 (represented by SALPT3), would become a basis variable for the optimal solution to our new LP problem.

REDCST OUTPUT

CASE ITERATION 4 OBJECTIVE VALUE 525.00000

LABEL	COST	REDUCED COST	LABEL	COST	REDUCED COST	LABEL	COST	REDUCED COST
MILMCH	.000000	2.250000	LATHE	.000000	.500000	SALPT3	.000000	1.250000
Z *H1	.000000	525.000000						

END REDCST OUTPUT

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2.3 Ranges

The inclusion of the card

```
RANGES
```

in our deck after the REDCST command will effect the <u>Dual Range Output</u> and the <u>Primal Range Output</u>.

2.3.1 Dual Range Output

Suppose the restraint (1) of our LP problem is in the form

$$\begin{aligned}
a_{11}\,x_1 + a_{12}\,x_2 + \cdots + a_{1n}\,x_n &\le b_1 \\
a_{21}\,x_1 + a_{22}\,x_2 + \cdots + a_{2n}\,x_n &\le b_2 \\
&\vdots \\
a_{m1}\,x_1 + a_{m2}\,x_2 + \cdots + a_{mn}\,x_n &\le b_m
\end{aligned}$$

We create a system of equations by introducing slack variables to obtain

$$\begin{aligned}
a_{11}\,x_1 + a_{12}\,x_2 + \cdots + a_{1n}\,x_n + x_{n+1} \phantom{+ x_{n+2} + x_{n+m}} &= b_1 \\
a_{21}\,x_1 + a_{22}\,x_2 + \cdots + a_{2n}\,x_n \phantom{+ x_{n+1}} + x_{n+2} \phantom{+ x_{n+m}} &= b_2 \\
&\vdots \\
a_{m1}\,x_1 + a_{m2}\,x_2 + \cdots + a_{mn}\,x_n \phantom{+ x_{n+1} + x_{n+2}} + x_{n+m} &= b_m
\end{aligned}$$

The dual range output gives us information about the nonbasis variables for the optimal solution to our LP problem. That is, it tells us the range in which the "original activities" of the non-basis variables may vary before the respective nonbasis variable will replace a listed basis variable.

By the "original activity" of a variable, we mean the value which that variable assumes in the first step of the simplex procedure. A slack variable will be in the basis for this first step of the simplex method. The original activity of a slack variable will therefore be the right-hand side of the inequality which it represents.

* * *From the dual range output, therefore, we obtain information about the right-hand sides of the inequalities which give rise to slack variables which are <u>not</u> in the basis for the optimal solution.* * *

The dual range output answers the following question:

* * Suppose that x_j is a slack variable which is <u>not</u> in the basis for the optimal solution to our problem. By what amounts may the right-hand side of the inequality which gives rise to this slack variable vary before a basis variable for the optimal solution to our original problem is replaced by x_j as a basis variable to the optimal solution of our new problem? * *

x_4, x_5, and x_7 (represented by the labels MILMCH, LATHE, and SALPT3, respectively) are the variables which are not in the basis for the optimal solution to our example problem. Furthermore, x_4, x_5, and x_7 are slack variables corresponding respectively to the first, second, and fourth inequalities of condition (1). The "original activities" of x_4, x_5, and x_7 are the right-hand sides of these three inequalities: 200, 100, and 20, respectively.

The dual range output for our example is given on the following page. From this we read that the right hand side of inequality MILMCH may vary between 200-73.333333 = 126.666667 and 200 + 73.333333 = 273.333333 without effecting a change in the basis variables for the optimal solution. If the right-hand side of this inequality is decreased beyond 126.666667 in our new LP problem, x_1 (represented by PROD 1), which is a basis variable for our original problem, will no longer be a basis variable for the optimal solution of our new LP problem. If the right-hand side of this inequality is increased beyond 273.333333, the basis variable x_6 (represented by GRNDER) will no longer be a basis variable for the optimal solution of our new LP problem.

Likewise, if the right-hand side of inequality LATHE decreases beyond 100 - 10 = 90, basis variable x_6 (represented by GRNDER) will leave the basis. If the right-hand side of this inequality increases beyond 100 + 110 = 210, basis variable x_1 (represented by PROD 1) will no longer be in the basis for the optimal solution to our new problem.

Also, if the right-hand side of inequality SALPT3 decreases beyond 20 - 20 = 0, basis variable x_2 (represented by PROD 2) will not be in the basis for the optimal solution to our new problem. If the right-hand side of this inequality increases beyond 20 + 24.444444 = 44.444444, x_1, a basis variable for the optimal solution to our original LP problem, will not be in the basis for the optimal solution to our new LP problem.

DUAL RANGE OUTPUT

CASE ITERATION 4 OBJECTIVE VALUE 525.00000

		- - - LIMITS	OF RANGE - - -		
LABEL	ORIG. ACT.	LABEL	INCREMENT	LABEL	INCREMENT
MILMCH	200.000000	PROD 1	-73.333333	GRIDER	6.666667
LATHE	100.000000	GRIDER	-10.000000	PROD 1	110.000000
SALE13	20.000000	PROD 2	-20.000000	PROD 1	24.444444

END DUAL RANGE OUTPUT

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2.3.2 Primal Range Output

Let the objective function of our original LP problem be

$$c_1 x_1 + c_2 x_2 + . . . + c_n x_n$$

The primal range output answers the following question about the basis for the optimal solution to this original LP problem and the basis for the optimal solution to the LP problem created by changing the above objective function:

* * Suppose that x_j is a basis variable for the optimal solution to our original LP problem. By what amounts may c_j (the coefficient of x_j in the objective function) vary before x_j is replaced by another variable in the basis for the optimal solution to our new LP problem ? * *

The primal range output for our example problem is given on the following page. From this we read that if 0, the objective function coefficient of x_6 (represented by GRNDER), decreases beyond -2, x_5 (represented by LATHE), rather than x_6, will be in the basis for the optimal solution to the new LP problem.

Also, if 6, the objective function coefficient of x_2, decreases beyond 5 or increases beyond 15, x_2 will be replaced by x_5 or x_4, respectively, in the basis for the optimal solution to the new LP problem.

Finally, if the objective function coefficient of x_2, decreases beyond 8 - 1.25 = 6.75, x_7, rather than x_3, will be in

PRIMAL RANGE OUTPUT

CASE ITERATION 4 OBJECTIVE VALUE 525.00000

		- - - - - - LIMITS OF RANGE	- - - - - -		
LABEL	COST	LABEL	INCREMENT	LABEL	INCREMENT
PROFIT	.000000		-9999.000000	MILMCH	1.000000
GRNDER	.000000	LATHE	-2.000000	MILMCH	6.000000
PROD 1	20.000000	MILMCH	-12.000000	SALPT3	2.222222
PROD 2	6.000000	LATHE	-1.000000	MILMCH	9.000000
PROD 3	8.000000	SALPT3	-1.250000		9999.000000

END PRIMAL RANGE OUTPUT

the basis for the optimal solution to our new LP problem. However, this coefficient of x_3 may increase as much as we desire (indicated by the 9999) without causing x_3 to leave the optimal basis for the resulting LP problem.

We have seen that if we have an LP problem:

Maximize $c_1 x_1 + c_2 x_2 + \dots + c_n x_n$

subject to:

$$
\begin{aligned}
(1)\quad & a_{11} x_1 + a_{12} x_2 + \dots + a_{1n} x_n = b_1 \\
& a_{21} x_1 + a_{22} x_2 + \dots + a_{2n} x_n = b_2 \\
& \vdots \\
& a_{m1} x_1 + a_{m2} x_2 + \dots + a_{mn} x_n = b_m
\end{aligned}
$$

$$(2)\quad x_i \geq 0; \quad i = 1, 2, \dots, n$$

for which we have a solution, and if we then alter a c_j or a b_j slightly to obtain a new LP problem, we may be able to determine from the output of our original problem, the basis for the optimal solution to the new problem. Knowing the basis for the optimal solution to this new problem enables us to find this optimal solution in a straightforward manner rather than attacking the problem with no information about the solution.

If we are solving the problem by hand, we merely solve $DX = B$ to obtain $X = D^{-1}B$ where D is the square $m \times m$ matrix whose columns are the optimal basis vectors.

If we are solving the problem on the UNIVAC 1108, we can specify the basis variables in our deck of cards. For the specific technique here, refer to UP-4138, Section 5, page 54, 5.29.1.2.

2.4 Errors

The card

1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26	27	28	29	30	31	32	33	34	35	36	37	38	39	40
E	R	R	O	R	S																																		

included in our deck after the command RANGES effects the output of the Dual Error Analysis and the Primal Error Analysis. These errors may arise as a result of the repeated rounding of decimals at each iteration.

2.4.1 Dual Error Analysis

Recall that in each tableau of the simplex method, we have a vector, T, whose components are the objective function coefficients of the variables which are in the basis for this tableau.

The optimal basis gives rise to our final tableau. The T vector for this final tableau is then $\begin{pmatrix} c_s \\ c_r \\ \vdots \\ c_t \end{pmatrix}$ where $x_s, x_r, \ldots x_t$ are the basis variables for the optimal solution to the LP problem.

For this T, we can then calculate $T \cdot P_j - c_j$ for $j = 1, 2, \ldots, n$.

Note that if x_s is a basis variable, then $P_q = e_q = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{pmatrix}$ (q-1 zeros) in the final tableau. Hence $T \cdot P_q = 0c_1 + 0c_2 + \ldots + 1c_q + \ldots + 0c_n = c_q$ for a basis vector P_q.

If we let $Z_j = T \cdot P_j$, we see that if our calculations are exact, we should have $Z_j = c_j$ for those j for which x_j is an optimal basis vector.

The dual error analysis output lists the labels for the optimal basis variables and then the corresponding Z_j and c_j and their difference which should be zero except for rounding errors.

The dual error analysis for our example is given on the following page.

2.4.2 Primal Error Analysis

An error analysis whose nature is easier to understand is the primal error analysis. We look at condition (1) to our LP problem in its final form as a system of simultaneous linear equations:

$$\begin{aligned} a_{11}\, x_1 + a_{12}\, x_2 + \cdot\cdot\cdot\cdot + a_{1n}\, x_n &= b_1 \\ a_{21}\, x_1 + a_{22}\, x_2 + \cdot\cdot\cdot\cdot + a_{2n}\, x_n &= b_2 \\ &\vdots \\ a_{m1}\, x_1 + a_{m2}\, x_2 + \cdot\cdot\cdot\cdot + a_{mn}\, x_n &= b_m \end{aligned}$$

We take the optimal solution, $X = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}$, which we obtain and substitute these values into each of the m equations. We then compare the value obtained by calculating

$$a_{i1}\, x_1 + a_{i2}\, x_2 + a_{i3}\, x_3 \cdot \cdot \cdot + a_{in}\, x_n$$

20 AUG 70 .001.022.001

DUAL ERROR ANALYSIS

LABEL	Z(J)	C(J)	ERROR
PROFIT	.000	.000	.00000000
GRNDER	.000	.000	.00000000
PROD 1	20.000	20.000	.00000000
PROD 2	6.000	6.000	.00000000
PROD 3	8.000	8.000	.00000000

MAXIMUM DUAL ERROR OF .00000000 FOR VARIABLE

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(for $x_1, x_2, \ldots, x_n$ of the optimal solution) with the b_i on the right-hand side of the equation.

Our primal error analysis output, then, contains the name created for each of the equations with the calculated value of $a_{i1} x_1 + a_{i2} x_2 + \ldots + a_{in} x_n$, the b_i on the right-hand side of the designated equation, and the difference (which should be zero except for rounding errors).

The primal error analysis for our example problem is given on the following page.

20 AUG 70 .001.023.001

PRIMAL ERROR ANALYSIS

LABEL	ORIG B(I)	CALC B(I)	ERROR
PROFIT	.000	.000	.00000000
MILMCH	200.000	200.000	.00000000
LATHE	100.000	100.000	.00000000
GRNDER	50.000	50.000	.00000000
SALPT3	20.000	20.000	.00000000

MAXIMUM PRIMAL ERROR OF .00000000 FOR VARIABLE

CHAPTER 3: SAMPLE RUNS

We shall list here a printing of the deck of cards necessary to obtain the information which we have discussed as well as the information obtained as a result of submitting said deck of cards. We shall do this for the five problems discussed in Chapter 2, Part II.

3.1 The Product Manufacturing Problem

We have created names for the variables x_1, x_2, x_3, the four inequalities, and the objective function in Section 1.3, Part III. The 1108 then assigns the four names which we have created for the four inequalities to the four respective slack variables x_4, x_5, x_6, and x_7.

Our deck of cards for this problem then has the following order:

```
@ RUN,/R     RCTPR1,NHQRCT,LPPWR,15,50
@FREE TPF$
@ASG,T TPF$,F///500
@ASG,T LPTAPE,T,1815
@ASG,T 9,T,SCRTCH
@FIND,A LPTAPE.LP1108/E8S
@COPIN,A LPTAPE.LP1108/E8S,TPF$.
@FREE LPTAPE
@XQT .LP1108/E8S
LOAD
ROW ID
     5       PROFIT      +MILMCH      +LATHE       +GRNDER      +SALPT3
MATRIX
      PROD 1PROFIT   20.
      PROD 1MILMCH    8.
      PROD 1LATHE     4.
      PROD 1GRNDER    2.
      PROD 2PROFIT    6.
      PROD 2MILMCH    2.
      PROD 2LATHE     3.
      PROD 3PROFIT    8.
      PROD 3MILMCH    3.
      PROD 3GRNDER    1.
      PROD 3SALPT3    1.
FIRSTB
             MILMCH  200.
```

```
              LATHE    100.
              GRNDER    50.
              SALPT3    20.
ENDATA

MATRIX
EQLIST
MTXMAP
GOGOGO
PRIMAL
DUAL
REDCST
RANGES
ERRORS
ENDJOB
& FIN
```

We have seen in Chapters 1 and 2 the information which we obtain as a result of submitting these cards.

3. The Diet Problem

Recall from Section 2.2, Part II, we wish to minimize

$$1.0x_1 + 1.1x_2 + 0.5x_3 \qquad \text{(COST)}$$

subject to the restraints:

(1) $x_1 + x_2 + 10x_3 \geq 1$ (VIT A)

$100x_1 + 10x_2 + 10x_3 \geq 50$ (VIT C)

$10x_1 + 100x_2 + 10x_3 \geq 10$ (VIT D)

(2) $x_i \geq 0; \quad i = 1, 2, 3$

We create the following names for the variables x_1, x_2, x_3:

GLMILK = x_1 = the number of gallons of milk in the daily diet

LBBEEF = x_2 = the number of pounds of beef in the daily diet

DZEGGS = x_3 = the number of dozens of eggs in the daily diet

We have created names for the three restraints of condition (1) and the objective function and have labeled these rows accordingly. The 1108 will assign the label which we have given to any inequality to the slack variable which arises from this inequality.

Therefore, if we write condition (1) in the form of three linear <u>equations</u>, we have:

$$\begin{array}{rrrlll} -x_1 & - \; x_2 & - \; 10x_3 & + \; x_4 & & = -1 \\ -100x_1 & -10x_2 & - \; 10x_3 & + \; x_5 & & = -50 \\ -10x_1 & -100x_2 & - \; 10x_3 & & + \; x_6 & = -10 \end{array}$$

(See Chapter 1, Part II)

The 1108 will assign: VIT A = x_4; VIT C = x_5; VIT D = x_6.

A listing of the cards for this problem follows.

```
@ RUN,/R     RCTPB2,NHGRCT,LPPWB,15,50
@FREE TPF$
@ASG,T TPF$,F///500
@ASG,T LPTAPE,T,1815
@ASG,T 9,T,SCRTCH
@FIND,A LPTAPE.LP1108/E8S
@COPIN,A LPTAPE.LP1108/E8S,TPF$.
@FREE LPTAPE
@XQT .LP1108/E8S
SET OBJECT TO (MIN    )
LOAD
ROW ID
     4       COST        -VIT  A      -VIT  C      -VIT  D
MATRIX
      GLMILKCOST       1.
      GLMILKVIT  A     1.
      GLMILKVIT  C   100.
      GLMILKVIT  D    10.
      LBBEEFCOST       1.1
      LBBEEFVIT  A     1.
      LBBEEFVIT  C    10.
      LBBEEFVIT  D   100.
      DZEGGSCOST        .5
      DZEGGSVIT  A    10.
      DZEGGSVIT  C    10.
      DZEGGSVIT  D    10.
```

```
FIRSTB
              VIT  A    1.
              VIT  C   50.
              VIT  D   10.
ENDATA

MATRIX
EGLIST
MTXMAP
GOGOGO
PRIMAL
DUAL
REDCST
RANGES
ERRORS
ENDJOB
G FIN
```

As a result, we obtain the information given on the following pages. Note that page 170 relates to the MTXMAP output given on page 171.

20 AUG 70 .001.004.001

CASE MATRIX TABLEAU

	LABEL	COST	B-VEC	GLMILK	LBBEEF	DZEGGS	*B1
	COST	.000000	.000000	-1.000000	-1.100000	-.500000	.000000
ROW LABEL							
1 COST		.000000	.000000	1.000000	1.100000	.500000	.000000
2 VIT A		.000000	-1.000000	-1.000000	-1.000000	-10.000000	-1.000000
3 VIT C		.000000	-50.000000	-100.000000	-10.000000	-10.000000	-50.000000
4 VIT D		.000000	-10.000000	-10.000000	-100.000000	-10.000000	-10.000000

END OF MATRIX TABLEAU

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```
20 AUG 70                                                                                   .001.006.001

CASE.                          ENLIST OUTPUT

EQUATION     1 LABEL COST   COST =      .000000 ORIGINAL BI =      .000000

+   1.00000(COST   ) +   1.00000(GLMILK) +   1.10000(LBBEEF) +    .50000(DZEGGS)

EQUATION     2 LABEL VIT  A COST =      .000000 ORIGINAL BI =   -1.000000

+   1.00000(VIT  A) -   1.00000(GLMILK) -   1.00000(LBBEEF) -  10.00000(DZEGGS) -   1.00000(*BI    )

EQUATION     3 LABEL VIT  C COST =      .000000 ORIGINAL BI =  -50.000000

+   1.00000(VIT  C) - 100.00000(GLMILK) -  10.00000(LBBEEF) -  10.00000(DZEGGS) -  50.00000(*BI    )

EQUATION     4 LABEL VIT  D COST =      .000000 ORIGINAL BI =  -10.000000

+   1.00000(VIT  D) -  10.00000(GLMILK) - 100.00000(LBBEEF) -  10.00000(DZEGGS) -  10.00000(*BI    )

END ENLIST OUTPUT
```

```
20 AUG 70                                              .001.009.001

              C B G L D .
              O - L B Z B
              S V M B E 1
              T L I E G
                C L E G
                  K F S

      COST    . .-1-6-5 .
    COST.     . . 1 6 5 .
    VIT  A    .-1-1-1-6-1
    VIT  C    .-/-7-6-6-7
    VIT  D    .-6-6-7-6-6
```

THE MATRIX ELEMENTS ARE REPRESENTED AS FOLLOWS

CODE	MAGNITUDE OF ELEMENTS GREATER THAN	EQUAL TO OR LESS THAN	OCCURENCES
0	0.0	0.0001	0
2	0.0001	0.001	0
3	0.001	0.01	0
4	0.01	0.1	0
5	0.1	0.99999999	2
1	=1.0		6
6	1.0	10.0	9
7	10.0	100.0	4
8	100.0	1000.0	0
9	1000.0		0

END MTXMAP OUTPUT

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20 AUG 70 .001.013.001

PRIMAL OUTPUT

CASE ITERATION 5 OBJECTIVE VALUE -.56481481

	LABEL	COST	ACTIVITY	LABEL	COST	ACTIVITY	LABEL	COST	ACTIVITY
E	COST	.000000	-.5648148+00	GLMILK	-1.000000	.490741	LBBEEF	-1.100000	.046296
	DZEGGS	-.500000	.046296						

END PRIMAL OUTPUT

Note that the 1108 has changed our problem to the following:

Maximize

$$- 1.0x_1 - 1.1x_2 - 0.5x_3$$

subject to:

$$(1) \quad \begin{aligned} -x_1 - x_2 - 10x_3 &\leq -1 \\ -100x_1 - 10x_2 - 10x_3 &\leq -50 \\ -10x_1 - 100x_2 - 10x_3 &\leq -10 \end{aligned}$$

$$(2) \quad x_i \geq 0; \quad i = 1, 2, 3$$

The dual output, reduced cost output, etc., for this example are given on the following pages.

DUAL OUTPUT

CASE ITERATION 5 OBJECTIVE VALUE -.56481481

	LABEL	COST	SHADOW PRICE	LABEL	COST	SHADOW PRICE	LABEL	COST	SHADOW PRICE
E	COST	.000000	.000000	VIT A	.000000	.031481	VIT C	.000000	.008704
	VIT D	.000000	.009815						

END DUAL OUTPUT

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20 AUG 70 .001.017.001

REDCST OUTPUT

CASE ITERATION 5 OBJECTIVE VALUE =.56481481

LABEL	COST	REDUCED COST	LABEL	COST	REDUCED COST	LABEL	COST	REDUCED COST
VIT A	.000000	.031431	VIT C	.000000	.008704	VIT D	.000000	.009815
7 +R1	.000000	-.564815						

END REDCST OUTPUT

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20 AUG 70 .001.019.001

DUAL RANGE OUTPUT

CASE ITERATION 5 OBJECTIVE VALUE -.56481481

- - - - - - LIMITS OF RANGE - - - - - -

LABEL	ORIG. ACT.	LABEL	INCREMENT	LABEL	INCREMENT
VIT A	-1.000000	LBBEEF	-4.999999	DZEGGS	.454545
VIT C	-50.000000	DZEGGS	-49.999999	GLMILK	48.181819
VIT D	-10.000000	DZEGGS	-50.000000	LBBEEF	4.545454

END DUAL RANGE OUTPUT

PRIMAL RANGE OUTPUT

CASE ITERATION 5 OBJECTIVE VALUE -.56481431

- - - - - - LIMITS OF RANGE - - - - - -

LABEL	COST	LABEL	INCREMENT	LABEL	INCREMENT
COST	.000000	VIT A	-1.000000		9999.000000
05MILK	-1.000000	VIT A	-3.400003	VIT C	.854545
06BEEF	-1.100000	VIT A	-3.400001	VIT D	.963636
07EGGS	-.500000	VIT C	-9.400000	VIT A	.309091

END PRIMAL RANGE OUTPUT

DUAL ERROR ANALYSIS .

LABEL	Z(J)	C(J)	ERROR
COST	.000	.000	.00000000
GLMILK	-1.000	-1.000	.00000000
LBBEEF	-1.100	-1.100	.14901161-07
DZEGGS	-.500	-.500	-.74505806-07

MAXIMUM DUAL ERROR OF -.74505806-07 FOR VARIABLE DZEGGS

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20 70 .001.023.001

PRIMAL ERROR ANALYSIS.

LABEL	ORIG B(I)	CALC B(I)	ERROR
COST	.000	.000	-.12572855-07
VIT A	-1.000	-1.000	.00000000
VIT C	-50.000	-50.000	.47683716-06
VIT D	-10.000	-10.000	-.11920929-06

MAXIMUM PRIMAL ERROR OF .47683716-06 FOR VARIABLE VIT C

3.3 The Gasoline Mix Problem

Recall from 2.3, Part II, that we wish to maximize

$$2.5x_1 - 0.5x_2 + 1.5x_3 + 0.5x_4 + 1.5x_5 - 1.5x_6 + 0.5x_7 - 0.5x_8 + 0.5x_9$$
$$-2.5x_{10} - 0.5x_{11} - 1.5x_{12}$$

subject to:

(1)

$$\begin{array}{llllllllllllll}
x_1 & & & & +x_5 & & & & +x_9 & & & & \leq & 3000\,(\text{INGRD1}) \\
& x_2 & & & & +x_6 & & & & +x_{10} & & & \leq & 2000\,(\text{INGRD2}) \\
& & x_3 & & & & +x_7 & & & & +x_{11} & & \leq & 4000\,(\text{INGRD3}) \\
& & & x_4 & & & & +x_8 & & & & +x_{12} & \leq & 1000\,(\text{INGRD4}) \\
0.7x_1 & -\ 0.3x_2 & -\ 0.3x_3 & -\ 0.4x_4 & & & & & & & & & \leq & 0\,(\text{PCT 1A}) \\
-0.4x_1 & +\ 0.6x_2 & -\ 0.4x_3 & -\ 0.4x_4 & & & & & & & & & \geq & 0\ (\text{PCT 2A}) \\
-0.5x_1 & -\ 0.5x_2 & +\ 0.5x_3 & -\ 0.5x_4 & & & & & & & & & \leq & 0\,(\text{PCT 3A}) \\
& & & & 0.5x_5 & -\ 0.5x_6 & -\ 0.5x_7 & -\ 0.5x_8 & & & & & \leq & 0\,(\text{PCT 1B}) \\
& & & & -0.1x_6 & +\ 0.9x_6 & -\ 0.1x_7 & -\ 0.1x_8 & & & & & \geq & 0\,(\text{PCT 2B}) \\
& & & & & & & & 0.3x_9 & -\ 0.7x_{10} & -\ 0.7x_{11} & -\ 0.7x_{12} & \leq & 0\,(\text{PCT 1C})
\end{array}$$

(2) $x_i \geq 0; \quad i = 1, 2, \ldots, 12$

Here we have again indicated the name given to each restraint of condition (1) and to the objective function. We also create the following names for the variables $x_1, x_2, \ldots, x_{12}$. We let:

1 IN A=x_1= # barrels of constituent 1 allocated to gasoline grade A per day
2 IN A=x_2= # " " " 2 " " " grade A " "
3 IN A=x_3= # " " " 3 " " " grade A " "
4 IN A=x_4= # " " " 4 " " " grade A " "
1 IN B=x_5= # barrels of constituent 1 allocated to gasoline grade B per day
2 IN B=x_6= # " " " 2 " " " grade B " "
3 IN B=x_7= # " " " 3 " " " grade B " "
4 IN B=x_8= # " " " 4 " " " grade B " "
1 IN C=x_9= # barrels of constituent 1 allocated to gasoline grade C per day
2 IN C=x_{10} # " " " 2 " " " grade C " "
3 IN C=x_{11} # " " " 3 " " " grade C " "
4 IN C=x_{12} # " " " 4 " " " grade C " "

A listing of the cards which we create to submit to the UNIVAC 1108 follows:

```
@ RUN,/R    RCTPR3,MHGRCT,LPPWR,15,50
@FREE TPF$
@ASG,T TPF$,F///500
@ASG,T LPTAPE,T,1815
@ASG,T 9,T,SCRTCH
@FIND,A LPTAPE.LP1108/E8S
@CCPIN,A LPTAPE.LP1108/E8S,TPF$.
@FREE LPTAPE
@XQT .LP1108/E8S
LOAD
ROW ID
     5      PROFIT    +INGRD1    +INGRD2    +INGRD3    +INGRD4
     5     +PCT 1A    -PCT 2A    +PCT 3A    +PCT 1B    -PCT 2B
     1     +PCT 1C
```

```
MATRIX
       1 IN APROFIT    2.5
       1 IN AINGRD1    1.0
       1 IN APCT 1A     .7
       1 IN APCT 2A-    .4
       1 IN APCT 3A-    .5
       2 IN APROFIT-    .5
       2 IN AINGRD2    1.
       2 IN APCT 1A-    .3
       2 IN APCT 2A     .6
       2 IN APCT 3A-    .5
       3 IN APROFIT    1.5
       3 IN APCT 1A-    .3
       3 IN APCT 2A-    .4
       3 IN APCT 3A     .5
       4 IN APROFIT     .5
       4 IN AINGRD4    1.0
       4 IN APCT 1A-    .3
       4 IN APCT 2A-    .4
       4 IN APCT 3A-    .5
       1 IN BPROFIT    1.5
       1 IN BINGRD1    1.
       1 IN BPCT 1B     .5
       1 IN BPCT 2B-    .1
       2 IN BPROFIT-   1.5
       2 IN BINGRD2    1.0
       2 IN BPCT 1B-    .5
       2 IN BPCT 2B     .9
       3 IN BPROFIT     .5
       3 IN BINGRD3    1.0
       3 IN BPCT 1B-    .5
       3 IN BPCT 2B-    .1
       4 IN BPROFIT-    .5
       4 IN BINGRD4    1.0
       4 IN BPCT 1B-    .5
       4 IN BPCT 2B-    .1
       1 IN CPROFIT     .5
       1 IN CINGRD1    1.0
       1 IN CPCT 1C     .3
       2 IN CPROFIT-   2.5
       2 IN CINGRD2    1.0
       2 IN CPCT 1C-    .7
       3 IN CPROFIT-    .5
       3 IN CINGRD3    1.0
       3 IN CPCT 1C-    .7
       4 IN CPROFIT-   1.5
       4 IN CINGRD4    1.0
       4 IN CPCT 1C-    .7
```

```
FIRSTR
             INGRD1 3000.
             INGRD2 2000.
             INGRD3 4000.
             INGRD4 1000.
ENDATA
MATRIX
EGLIST
MTXMAP
GOGOGO
PRIMAL
DUAL
REDCST
RANGES
ERRORS
ENDJOB
& FIN
```

As a result, we obtain the information given on the following 13 pages.

20 AUG 70 .001.004.001

CASE MATRIX TABLEAU

LABEL	COST	B-VEC	1 IN A	2 IN A	3 IN A	4 IN A	1 IN B	2 IN B	3 IN B
COST	.000000	.000000	2.500000	-.500000	1.500000	.500000	1.500000	-1.500000	.500000
ROW LABEL									
1 PROFIT	.000000	.000000	2.500000	-.500000	1.500000	.500000	1.500000	-1.500000	.500000
2 INGRD1	.000000	1000.000000	1.000000	.000000	.000000	.000000	1.000000	.000000	.000000
3 INGRD2	.000000	2000.000000	.000000	1.000000	.000000	.000000	.000000	1.000000	.000000
4 INGRD3	.000000	4000.000000	.000000	.000000	.000000	.000000	.000000	.000000	1.000000
5 INGRD4	.000000	1000.000000	.000000	.000000	.000000	1.000000	.000000	.000000	.000000
6 PCT 1A	.000000	.000000	.700000	-.300000	-.300000	-.300000	.000000	.000000	.000000
7 PCT 2A	.000000	.000000	.400000	-.600000	.400000	.400000	.000000	.000000	.000000
8 PCT 3A	.000000	.000000	-.500000	-.500000	.500000	-.500000	.000000	.000000	.000000
9 PCT 1B	.000000	.000000	.000000	.000000	.000000	.000000	.500000	-.500000	-.500000
10 PCT 2B	.000000	.000000	.000000	.000000	.000000	.000000	.100000	-.900000	.100000
11 PCT 1C	.000000	.000000	.000000	.000000	.000000	.000000	.000000	.000000	.000000

.001.005.001

20 AUG 70

CASE

MATRIX TABLEAU

LABEL	4 IN B	1 IN C	2 IN C	3 IN C	4 IN C	•BI
COST	-.500000	.500000	-2.500000	-.500000	-1.500000	.000000
ROW LABEL						
1 PROFIT	-.500000	.500000	-2.500000	-.500000	-1.500000	.000000
2 INGRD1	.000000	1.000000	.000000	.000000	.000000	3000.000000
3 INGRD2	.000000	.000000	1.000000	.000000	.000000	2000.000000
4 INGRD3	.000000	.000000	.000000	1.000000	.000000	4000.000000
5 INGRD4	1.000000	.000000	.000000	.000000	1.000000	1000.000000
6 PCT 1A	.000000	.000000	.000000	.000000	.000000	.000000
7 PCT 2A	.000000	.000000	.000000	.000000	.000000	.000000
8 PCT 3A	.000000	.000000	.000000	.000000	.000000	.000000
9 PCT 1B	-.500000	.000000	.000000	.000000	.000000	.000000
10 PCT 2B	.100000	.000000	.000000	.000000	.000000	.000000
11 PCT 1C	.000000	.300000	-.700000	-.700000	-.700000	.000000

END OF MATRIX TABLEAU

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```
20 AUG 70                                                                                        .001.007.001

CASE                           EQLIST OUTPUT

EQUATION    1 LABEL PROFIT COST =      .000000 ORIGINAL BI =      .000000

+   1.00000(PROFIT) +   2.50000(1 IN A) -    .50000(2 IN A) +   1.50000(3 IN A) +    .50000(4 IN A) +   1.50000(1 IN B)
-   1.50000(2 IN B) +    .50000(3 IN B) -    .50000(4 IN B) +    .50000(1 IN C) -   2.50000(2 IN C) -    .50000(3 IN C)
-   1.50000(4 IN C)

EQUATION    2 LABEL INGRD1 COST =      .000000 ORIGINAL BI = 3000.000000

+   1.00000(INGRD1) +   1.00000(1 IN A) +   1.00000(1 IN B) +   1.00000(1 IN C) +3000.00000(*BI    )

EQUATION    3 LABEL INGRD2 COST =      .000000 ORIGINAL BI = 2000.000000

+   1.00000(INGRD2) +   1.00000(2 IN A) +   1.00000(2 IN B) +   1.00000(2 IN C) +2000.00000(*BI    )

EQUATION    4 LABEL INGRD3 COST =      .000000 ORIGINAL BI = 4000.000000

+   1.00000(INGRD3) +   1.00000(3 IN B) +   1.00000(3 IN C) +4000.00000(*BI    )

EQUATION    5 LABEL INGRD4 COST =      .00000 ORIGINAL BI = 1000.000000

+   1.00000(INGRD4) +   1.00000(4 IN A) +   1.00000(4 IN B) +   1.00000(4 IN C) +1000.00000(*BI    )

EQUATION    6 LABEL PCT 1A COST =      .000000 ORIGINAL BI =      .000000

+   1.00000(PCT 1A) +    .70000(1 IN A) -    .30000(2 IN A) -    .30000(3 IN A) -    .30000(4 IN A)

EQUATION    7 LABEL PCT 2A COST =      .000000 ORIGINAL BI =      .000000

+   1.00000(PCT 2A) +    .40000(1 IN A) -    .60000(2 IN A) +    .40000(3 IN A) +    .40000(4 IN A)

EQUATION    8 LABEL PCT 3A COST =      .00000 ORIGINAL BI =      .000000

+   1.00000(PCT 3A) -    .50000(1 IN A) -    .50000(2 IN A) +    .50000(3 IN A) -    .50000(4 IN A)

EQUATION    9 LABEL PCT 1B COST =      .00000 ORIGINAL BI =      .000000

+   1.00000(PCT 1B) +    .50000(1 IN B) -    .50000(2 IN B) -    .50000(3 IN B) -    .50000(4 IN B)
```

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20 AUG 70                                                        .001.007.002

EQUATION   10 LABEL PCT 2B COST =     .000000 ORIGINAL BI =     .000000
 +   1.00000(PCT 2B) +    .10000(1 IN B) -    .90000(2 IN B) +    .10000(3 IN B) +    .10000(4 IN B)

EQUATION   11 LABEL PCT 1C COST =     .000000 ORIGINAL BI =     .000000
 +   1.00000(PCT 1C) +    .30000(1 IN C) -    .70000(2 IN C) -    .70000(3 IN C) -    .70000(4 IN C)

END ENLIST OUTPUT
```

20 AUG 70 .001.010.001

```
           C b 1 2 3 4 1 2 3 4 1 2 3 4 .
           U -                         B
           S V I I I I I I I I I I I I 1
           T E N N N N N N N N N N N N
             C
               A A A A B B B B C C C C

   COST    . . 6-5 6 5 6-6 5-5 5-6-5-6 .
 PROFIT    . . 6-5 6 5 6-6 5-5 5-6-5-6 .
 INGRD1    . 9 1 . . . 1 . . . 1 . . . 9
 INGRD2    . 9 . 1 . . . 1 . . . 1 . . 9
 INGRD3    . 9 . . . . . . 1 . . . 1 . 9
 INGRD4    . 6 . . . 1 . . . 1 . . . 1 8
 PCT 1A    . . 5-5-5-5 . . . . . . . . .
 PCT 2A    . . 5-5 5 5 . . . . . . . . .
 PCT 3A    . .-5-5 5-5 . . . . . . . . .
 PCT 1B    . . . . . . 5-5-5-5 . . . . .
 PCT 2B    . . . . . . 4-5 4 4 . . . . .
 PCT 1C    . . . . . . . . . . 5-5-5-5 .
```

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20 AUG 70

THE MATRIX ELEMENTS ARE REPRESENTED AS FOLLOWS

CODE	MAGNITUDE OF ELEMENTS GREATER EQUAL TO OR THAN	LESS THAN	OCCURENCES
0	0.0	0.0001	0
2	0.0001	0.001	0
3	0.001	0.01	0
4	0.01	0.1	3
5	0.1	0.99999999	33
1	=1.0		11
6	1.0	10.0	12
7	10.0	100.0	0
8	100.0	1000.0	2
9	1000.0		6

END MTXMAP OUTPUT

20 AUG 70 .001.014.001

PRIMAL OUTPUT

CASE ITERATION 12 OBJECTIVE VALUE 7428.5713

	LABEL	COST	ACTIVITY	LABEL	COST	ACTIVITY	LABEL	COST	ACTIVITY
E	PROFIT	.000000	-.7428571+04	INGR04	.000000	1000.000000	PCT 1A	.000000	628.571411
	PCT 1B	.000000	1928.571442	PCT 1C	.000000	.000000	1 IN A	2.500000	314.285713
	2 IN A	-.500000	1257.142853	3 IN A	1.500000	1571.428543	1 IN B	1.500000	2685.714294
	2 IN B	-1.500000	742.857155	3 IN B	.500000	4000.000000			

END PRIMAL OUTPUT

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1 AUG 70 .001.016.001

DUAL OUTPUT

CASE ITERATION 12 OBJECTIVE VALUE 7428.5713

	LABEL	COST	SHADOW PRICE	LABEL	COST	SHADOW PRICE	LABEL	COST	SHADOW PRICE
E	PROFIT	.000000	.000000	INGR01	.000000	1.142857	INGR02	.000000	1.714286
	INGR03	.000000	.142857	INGR04	.000000	.000000	PCT 1A	.000000	.000000
	PCT 2A	.000000	3.571429	PCT 3A	.000000	.142857	PCT 1B	.000000	.000000
	PCT 2B	.000000	3.571429	PCT 1C	.000000	.000000			

END DUAL OUTPUT

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20 AUG 70

REDCST OUTPUT

CASE ITERATION 12 OBJECTIVE VALUE 7428.5713

LABEL	COST	REDUCED COST	LABEL	COST	REDUCED COST	LABEL	COST	REDUCED COST
INGR01	.000000	1.142857	INGR02	.000000	1.714286	INGR03	.000000	.142857
PCT 2A	.000000	3.571429	PCT 3A	.000000	.142857	PCT 2D	.000000	3.571429
4 IN A	.500000	.057143	4 IN B	-.500000	.857143	1 IN C	.500000	.642857
2 IN C	-2.500000	4.214286	3 IN C	-.500000	.642857	4 IN C	-1.500000	1.500000
Z .B1	.000000	7428.571411						

END REDCST OUTPUT

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20 AUG 70 .001.020.001

DUAL RANGE OUTPUT

CASE ITERATION 12 OBJECTIVE VALUE 7426.5713

		- - - - - - LIMITS OF RANGE - - - - - -			
LABEL	ORIG. ACT.	LABEL	INCREMENT	LABEL	INCREMENT
INGRD1	3000.000000	1 IN B	-2611.111115	PCT 1B	2250.000000
INGRD2	2000.000000	PCT 1A	-1222.222198		9999.000000
INGRD3	4000.000000	PCT 1B	-1894.736862		9999.000000
PCT 2A	.000000	1 IN A	-249.444443	PCT 1A	1466.666611
PCT 3A	.000000	PCT 1A	-666.666649	1 IN A	305.555557
PCT 2B	.000000	3 IN A	-1099.999985	2 IN B	650.000000
4 IN A	.000000	PCT 1A	-666.666649	1 IN A	305.555561
4 IN B	.000000	PCT 1B	-1894.736862	INGRD4	1000.000000
1 IN C	.000000	PCT 1B	-2250.000000	PCT 1C	.000000
2 IN C	.000000	PCT 1C	.000000	PCT 1A	1222.222198
3 IN C	.000000	PCT 1C	.000000	PCT 1B	1894.736862
4 IN C	.000000	PCT 1C	.000000	INGRD4	1000.000000

END DUAL RANGE OUTPUT

20 AUG 70 .001.021.001

PRIMAL RANGE OUTPUT

CASE ITERATION- 12 OBJECTIVE VALUE 7426.5713

LABEL	COST	LABEL	INCREMENT	LABEL	INCREMENT
		- - - - - - LIMITS OF RANGE	- - - - - -		
PROFIT	.000000		-9999.000000	1 IN C	1.000000
INGRD4	.000000	4 IN A	-.857143		9999.000000
PCT 1A	.000000	PCT 3A	-.151515	4 IN A	.909091
PCT 1B	.000000	INGRD3	-.263158	PCT 3A	.312500
PCT 1C	.000000	1 IN C	-2.142857	3 IN C	.918367
1 IN A	2.500000	4 IN A	-.833333	PCT 3A	.138889
2 IN A	-.500000	INGRD2	-1.666667	INGRD3	1.250000
3 IN A	1.500000	PCT 3A	-.166667	INGRD3	1.000000
1 IN B	1.500000	PCT 3A	-.138889	4 IN A	.833333
2 IN B	-1.500000	INGRD3	-1.250000	PCT 2A	3.125000
3 IN B	.500000	INGRD3	-.142857		9999.000000

END PRIMAL RANGE OUTPUT

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DUAL ERROR ANALYSIS

LABEL	Z(J)	C(J)	ERROR
PROFIT	.000	.000	.00000000
INGRDA	.000	.000	.00000000
PCT IA	.000	.000	.00000000
PCT IB	.000	.000	.00000000
PCT IC	.000	.000	.00000000
1 IN A	2.500	2.500	-.59604645-07
2 IN A	-.500	-.500	-.14901161-07
3 IN A	1.500	1.500	.14901161-07
1 IN B	1.500	1.500	.00000000
2 IN B	-1.500	-1.500	.00000000
3 IN B	.500	.500	.00000000

MAXIMUM DUAL ERROR OF -.59604645-07 FOR VARIABLE 1 IN A

20 AUG 70 .001.024.001

PRIMAL ERROR ANALYSIS

LABEL	ORIG B(I)	CALC B(I)	ERROR
PROFIT	.000	-.000	.15258769-04
INGRD1	3000.000	3000.000	.00000000
INGRD2	2000.000	2000.000	.00000000
INGRD3	4000.000	4000.000	.00000000
INGRD4	1000.000	1000.000	.00000000
PCT 1A	.000	.000	-.38146973-05
PCT 2A	.000	-.000	.76293945-05
PCT 3A	.000	-.000	.76293945-05
PCT 1B	.000	.000	-.30517578-04
PCT 2B	.000	-.000	.11444092-04
PCT 1C	.000	.000	.00000000

MAXIMUM PRIMAL ERROR OF -.30517578-04 FOR VARIABLE PCT 1B

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3.4 The Farming Problem

Recall from 2.4, Part II, that we wish to maximize

$$400(x_1 + x_2 + x_3) + 300(x_4 + x_5 + x_6) + 100(x_7 + x_8 + x_9) \quad \text{(PROFIT)}$$

subject to:

(1)

$$\begin{array}{lllllllllll}
x_1 & & & +x_4 & & & +x_7 & & & \leq 400 & \text{(ACRGE1)} \\
 & x_2 & & & +x_5 & & & +x_8 & & \leq 600 & \text{(ACRGE2)} \\
 & & x_3 & & & +x_6 & & & +x_9 & \leq 300 & \text{(ACRGE3)} \\
5x_1 & & & +4x_4 & & & +3x_7 & & & \leq 1500 & \text{(WAYER1)} \\
 & 5x_2 & & & +4x_5 & & & +3x_8 & & \leq 2000 & \text{(WATER2)} \\
 & & 5x_3 & & & +4x_6 & & & +3x_9 & \leq 900 & \text{(WATER3)} \\
x_1 & + x_2 & + x_3 & & & & & & & \leq 700 & \text{(AC C A)} \\
 & & & x_4 & + x_5 & + x_6 & & & & \leq 800 & \text{(AC C B)} \\
 & & & & & & x_7 & + x_8 & + x_9 & \leq 300 & \text{(AC C C)} \\
3x_1 & - 2x_2 & & +3x_4 & - 2x_5 & & +3x_7 & - 2x_8 & & = 0 & \text{(WKRES1)} \\
 & x_2 & - 2x_3 & & +x_5 & - 2x_6 & & +x_8 & - 2x_9 & = 0 & \text{(WKRES2)}
\end{array}$$

(2) $x_i \geq 0; \quad i = 1, 2, \ldots, 9$

We have again labeled each of the restraints of condition (1) and the objective function. We create the following names for the variables $x_1, x_2, x_3, \ldots, x_9$:

A1 = x_1 = the number of acres at farm 1 devotes to crop A
A2 = x_2 = " " " " " farm 2 " " crop A
A3 = x_3 = " " " " " farm 3 " " crop A
B1 = x_4 = " " " " " farm 1 " " crop B
B2 = x_5 = " " " " " farm 2 " " crop B
B3 = x_6 = " " " " " farm 3 " " crop B
C1 = x_7 = the number of acres at farm 1 devoted to crop C
C2 = x_8 = " " " " " farm 2 " " crop C
C3 = x_9 = " " " " " farm 3 " " crop C

Our deck for this run, then, consists of the following cards:

```
@ RUN,/R     RCTPB4,NHQRCT,LPPWR,15,50
@FREE TPF$
@ASG,T TPF$,F///500
@ASG,T LPTAPE,T,1815
@ASG,T 9,T,SCRTCH
@FIND,A LPTAPE.LP1108/E8S
@CCPIN,A LPTAPE.LP1108/E8S,TPF$.
@FREE LPTAPE
@XQT .LP1108/E8S
LOAD
ROW ID
     5      PROFIT     +ACRGE1     +ACRGE2     +ACRGE3     +WATER1
     5     +WATER2     +WATER3     +AC  C  A   +AC  C  B   +AC  C  C
     2      WKRES1      WKRES2
MATRIX
      A1    PROFIT   400.
      A1    ACRGE1     1.
      A1    WATER1     5.
      A1    AC  C  A   1.
      A1    WKRES1     3.
```

```
ENDATA
MATRIX
EGLIST
MTXMAP
GOGOGO
PRIMAL
DUAL
REDCST
RANGES
ERRORS
ENDJOB
& FIN
```

We obtain the information given on the following 13 pages.

3.5 The Nut Mix Problem

From Section 2.5, Part II, we wish to maximize

$$.15x_1 + .25x_2 + .15x_3 - .30x_4 + .10x_5 + 0x_6 - .40x_7 + 0x_8 - .10x_9 \text{ (PROFIT)}$$

(same as on page 93) subject to:

(1)

$$\begin{array}{lllllllllll}
-0.5\,x_1 & + 0.5\,x_2 & + 0.5\,x_3 & & & & & & & \leq & 0 \text{ (MIX A1)} \\
-0.25x_1 & + 0.75x_2 & - 0.25x_3 & & & & & & & \leq & 0 \text{ (MIX A2)} \\
 & & & -0.75x_4 & + 0.25x_5 & + 0.25x_6 & & & & \leq & 0 \text{ (MIX B1)} \\
 & & & -0.5\,x_4 & + 0.5\,x_5 & - 0.5\,x_6 & & & & \leq & 0 \text{ (MIX B2)} \\
x_1 & & & +x_4 & & & +x_7 & & & \leq & 100 \text{ (CAP C)} \\
 & x_2 & & & +x_5 & & & +x_8 & & \leq & 100 \text{ (CAP P)} \\
 & & x_3 & & & +x_6 & & & +x_9 & \leq & 60 \text{ (CAP [illegible]H)}
\end{array}$$

(2) $x_i \geq 0; \quad i = 1, 2, \ldots, 9$

We have again indicated the labels for each of the restraints in condition (1) and the objective function. We also create

20 AUG 70 .001.004.001

CASE -MATRIX TABLEAU

LABEL	COST	B-VEC	A1	A2	A3	B1	B2	B3	C1
COST	.000000	.000000	400.000000	400.000000	400.000000	300.000000	300.000000	300.000000	100.000000
ROW LABEL									
1 PROFIT	.000000	.000000	400.000000	400.000000	400.000000	300.000000	300.000000	300.000000	100.000000
2 ACRGE1	.000000	400.000000	1.000000	.000000	.000000	1.000000	.000000	.000000	1.000000
3 ACRGE2	.000000	600.000000	.000000	1.000000	.000000	.000000	1.000000	.000000	.000000
4 ACRGE3	.000000	300.000000	.000000	.000000	1.000000	.000000	.000000	1.000000	.000000
5 WATER1	.000000	1500.000000	5.000000	.000000	.000000	4.000000	.000000	.000000	3.000000
6 WATER2	.000000	2000.000000	.000000	5.000000	.000000	.000000	4.000000	.000000	.000000
7 WATER3	.000000	900.000000	.000000	.000000	5.000000	.000000	.000000	4.000000	.000000
8 AC C A	.000000	700.000000	1.000000	1.000000	1.000000	.000000	.000000	.000000	.000000
9 AC C B	.000000	800.000000	.000000	.000000	.000000	1.000000	1.000000	1.000000	.000000
10 AC C C	.000000	100.000000	.000000	.000000	.000000	.000000	.000000	.000000	1.000000
11 WKRES1	.000000	.000000	3.000000	-2.000000	.000000	3.000000	-2.000000	.000000	3.000000
12 WKRES2	.000000	.000000	.000000	1.000000	-2.000000	.000000	1.000000	-2.000000	.000000

20 AUG 70 .001.005.001

CASE MATRIX TABLEAU

ROW LABEL	LABEL / COST	C2	C3	B1
	COST	100.000000	100.000000	.000000
1	PROFIT	100.000000	100.000000	.000000
2	ACRGE1	.000000	.000000	400.000000
3	ACRGE2	1.000000	.000000	600.000000
4	ACRGE3	.000000	1.000000	300.000000
5	WATER1	.000000	.000000	1500.000000
6	WATER2	3.000000	.000000	2000.000000
7	WATER3	.000000	3.000000	900.000000
8	AC C A	.000000	.000000	700.000000
9	AC C B	.000000	.000000	800.000000
10	AC C C	1.000000	1.000000	300.000000
11	WKRES1	-2.000000	.000000	.000000
12	WKRES2	1.000000	-2.000000	.000000

END OF MATRIX TABLEAU

```
CASE                    EWLIST OUTPUT

EQUATION   1 LABEL PROFIT COST =   .000000 ORIGINAL BI =   .000000
+  1.00000(PROFIT) + 400.00000(A1   ) + 400.00000(A2   ) + 400.00000(A3   ) + 300.00000(B1   ) + 300.00000(B2   )
+ 300.00000(B3   ) + 100.00000(C1   ) + 100.00000(C2   ) + 100.00000(C3   )

EQUATION   2 LABEL ACRGE1 COST =   .000000 ORIGINAL BI =  400.000000
+  1.00000(ACRGE1) +   1.00000(A1   ) +   1.00000(B1   ) +   1.00000(C1   ) + 400.00000(*B1   )

EQUATION   3 LABEL ACRGE2 COST =   .000000 ORIGINAL BI =  600.000000
+  1.00000(ACRGE2) +   1.00000(A2   ) +   1.00000(B2   ) +   1.00000(C2   ) + 600.00000(*B1   )

EQUATION   4 LABEL ACRGE3 COST =   .000000 ORIGINAL BI =  300.000000
+  1.00000(ACRGE3) +   1.00000(A3   ) +   1.00000(B3   ) +   1.00000(C3   ) + 300.00000(*B1   )

EQUATION   5 LABEL WATER1 COST =   .000000 ORIGINAL BI = 1500.000000
+  1.00000(WATER1) +   5.00000(A1   ) +   4.00000(B1   ) +   3.00000(C1   ) +1500.00000(*B1   )

EQUATION   6 LABEL WATER2 COST =   .000000 ORIGINAL BI = 2000.000000
+  1.00000(WATER2) +   5.00000(A2   ) +   4.00000(B2   ) +   3.00000(C2   ) +2000.00000(*B1   )

EQUATION   7 LABEL WATER3 COST =   .000000 ORIGINAL BI =  900.000000
+  1.00000(WATER3) +   5.00000(A3   ) +   4.00000(B3   ) +   3.00000(C3   ) + 900.00000(*B1   )

EQUATION   8 LABEL AC C A COST =   .000000 ORIGINAL BI =  700.000000
+  1.00000(AC C A) +   1.00000(A1   ) +   1.00000(A2   ) +   1.00000(A3   ) + 700.00000(*B1   )

EQUATION   9 LABEL AC C B COST =   .000000 ORIGINAL BI =  800.000000
+  1.00000(AC C B) +   1.00000(B1   ) +   1.00000(B2   ) +   1.00000(B3   ) + 800.00000(*B1   )
```

```
EQUATION   10 LABEL AC C C COST =      .000000 ORIGINAL BI =  300.000000
 +  1.00000(AC C C) +   1.00000(C1     ) +   1.00000(C2     ) +   1.00000(C3     ) + 300.00000(*B1    )

EQUATION   11 LABEL WKRES1 COST =      .000000 ORIGINAL BI =      .000000
 +  1.00000(WKRES1) +   1.00000(A1     ) -   2.00000(A2     ) +   3.00000(B1     ) -   2.00000(B2     ) +   3.00000(C1     )
 -  2.00000(C2     )

EQUATION   12 LABEL WKRES2 COST =      .000000 ORIGINAL BI =      .000000
 +  1.00000(WKRES2) +   1.00000(A2     ) -   2.00000(A3     ) +   1.00000(B2     ) -   2.00000(B3     ) +   1.00000(C2     )
 -  2.00000(C3     )

END EQLIST OUTPUT
```

```
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           C B A A A B B B C C C . .
           U - 1 2 3 1 2 3 1 2 3 B
           S V                   J
           T E
             C

    COST   . . 8 8 8 8 8 8 7 7 7 .
  PROFIT   . . 8 8 8 8 8 8 7 7 7 .
  ACRGE1   . 8 1 . . 1 . . 1 . . 8
  ACRGE2   . 8 . 1 . . 1 . . 1 . 8
  ACRGE3   . 8 . . 1 . . 1 . . 1 8
  WATER1   . 9 6 . . 6 . . 6 . . 9
  WATER2   . 9 . 6 . . 6 . . 6 . 9
  WATER3   . 8 . . 6 . . 6 . . 6 8
  AC C A   . 8 1 1 1 . . . . . . 8
  AC C B   . 8 . . . 1 1 1 . . . 8
  AC C C   . 8 . . . . . . 1 1 1 8
  WKRES1   . . 6-6 . 6-6 . 6-6 . .
  WKRES2   . . . 1-6 . 1-6 . 1-6 .
```

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20 AUG 70 .001.011.001

THE MATRIX ELEMENTS ARE REPRESENTED AS FOLLOWS

CODE	MAGNITUDE OF ELEMENTS GREATER EQUAL TO OR THAN	LESS THAN	OCCURENCES
0	0.0	0.0001	0
2	0.0001	0.001	0
3	0.001	0.01	0
4	0.01	0.1	0
5	0.1	0.99999999	0
1	=1.0		21
6	1.0	10.0	18
7	10.0	100.0	6
8	100.0	1000.0	25
9	1000.0		4

END MTXMAP OUTPUT

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20 AUG 70

PRIMAL OUTPUT

CASE ITERATION . 7 OBJECTIVE VALUE 342499.99

	LABEL	COST	ACTIVITY	LABEL	COST	ACTIVITY	LABEL	COST	ACTIVITY
E	PROFIT	.000000	-.3425000+06	ACRGE1	.000000	100.000006	ACRGE2	.000000	150.000004
	ACRGE3	.000000	75.000000	WATER1	.000000	.000000	AC C A	.000000	200.000021
	AC C B	.000000	324.999992	AC C C	.000000	300.000000	A1	400.000000	299.999992
	A2	400.000000	199.999989	B2	300.000000	250.000011	B3	300.000000	225.000000

END PRIMAL OUTPUT

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20 AUG 70 .001.016.001

DUAL OUTPUT

CASE ITERATION 7 OBJECTIVE VALUE 342499.99

	LABEL	COST	SHADOW PRICE		LABEL	COST	SHADOW PRICE		LABEL	COST	SHADOW PRICE
E	PROFIT	.000000	.000000		ACRGE1	.000000	.000000		ACRGE2	.000000	.000000
	ACRGE3	.000000	.000000		WATER1	.000000	.000000		WATER2	.000000	100.000000
	WATER3	.000000	158.333328		AC C A	.000000	.000000		AC C B	.000000	.000000
	AC C C	.000000	.000000	Z	WKRES1	.000000	133.333332	Z	WKRES2	.000000	166.666656

END DUAL OUTPUT

20 AUG 70 .001.018.001

REDCST OUTPUT

CASE ITERATION 7 OBJECTIVE VALUE 342499.99

LABEL	COST	REDUCED COST	LABEL	COST	REDUCED COST	LABEL	COST	REDUCED COST
WATER2	.000000	100.000000	WATER3	.000000	150.333328	Z WKRES1	.000000	133.333332
Z WKRES2	.000000	166.666656	A3	400.000000	58.333328	B1	300.000000	99.999996
C1	100.000000	300.000000	C2	100.000000	99.999992	C3	100.000000	41.666672
Z B1	.000000	************						

END REDCST OUTPUT

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20 AUG 70 .001.020.001

DUAL RANGE OUTPUT

CASE ITERATION 7 OBJECTIVE VALUE 342499.99

- - - - - - LIMITS OF RANGE - - - - - -

LABEL	ORIG. ACT.	LABEL	INCREMENT	LABEL	INCREMENT
WATER2	2000.000000	A2	-199.999994	AC C A	200.000027
WATER3	900.000000	B2	-100.000007	WATER1	.000000
A3	.000000	WATER1	.000000	AC C A	75.000009
B1	.000000	WATER1	.000000	A1	299.999992
C1	.000000	WATER1	.000000	A1	299.999992
C2	.000000	AC C B	-162.499998	B2	125.000007
C3	.000000	B2	-100.000007	WATER1	.000000

END DUAL RANGE OUTPUT

20 AUG 70 .001.021.001

PRIMAL RANGE OUTPUT

CASE ITERATION 7 OBJECTIVE VALUE 342499.99

LABEL	COST	LABEL	INCREMENT	LABEL	INCREMENT
		- - - - - - LIMITS OF RANGE - - - - - -			
PROFIT	.000000		-9997.000000	C2	1.000000
ACRGE1	.000000	C3	-125.000016	A3	174.999987
ACRGE2	.000000	C3	-83.333344	A3	116.666656
ACRGE3	.000000	C3	-166.666687	A3	233.333313
WATER1	.000000	C3	-25.000003	A3	34.999997
AC C A	.000000	A3	-21.874998	C3	25.000004
AC C B	.000000	C3	-23.009527	A3	15.555554
AC C C	.000000	C3	-41.666672		9999.000000
A1	400.000000	B1	-99.999996	C3	125.000016
A2	400.000000	C3	-20.833336	A3	29.166665
B2	300.000000	A3	-23.333332	C3	16.666669
B3	300.000000	A3	-46.666662		9999.000000

END PRIMAL RANGE OUTPUT

20 AUG 70 .001.023.001

DUAL ERROR ANALYSIS

LABEL	Z(J)	C(J)	ERROR
PROFIT	.000	.000	.00000000
ACRGE1	.000	.000	.00000000
ACRGE2	.000	.000	.00000000
ACRGE3	.000	.000	.00000000
WATER1	.000	.000	.00000000
AC C A	.000	.000	.00000000
AC C B	.000	.000	.00000000
AC C C	.000	.000	.00000000
A1	400.000	400.000	-.38146973-05
A2	400.000	400.000	-.76293945-05
B2	300.000	300.000	-.76293945-05
B3	300.000	300.000	.00000000

MAXIMUM DUAL ERROR OF -.76293945-05 FOR VARIABLE A2

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PRIMAL ERROR ANALYSIS

LABEL	ORIG B(I)	CALC B(I)	ERROR
PROFIT	.000	-.004	.39062500-02
ACRGE1	400.000	400.000	.38146973-05
ACRGE2	600.000	600.000	.00000000
ACRGE3	300.000	300.000	.00000000
WATER1	1500.000	1500.000	.45776367-04
WATER2	2000.000	2000.000	.15258789-04
WATER3	900.000	900.000	.00000000
AC C A	700.000	700.000	.00000000
AC C B	800.000	800.000	.00000000
AC C C	300.000	300.000	.00000000
WKRES1	.000	-.000	.22888184-04
WKRES2	.000	.000	.00000000

MAXIMUM PRIMAL ERROR OF .39062500-02 FOR VARIABLE PROFIT the following names for the variables $x_1, x_2, \ldots x_9$.

MA-CAS = x_1 = the number of cashews in mixture A

MA-PEA = x_2 = the number of peanuts in mixture A

MA-HAZ = x_3 = the number of hazel nuts in mixture A

MB-CAS = x_4 = the number of cashews in mixture A

MB-PEA = x_5 = the number of peanuts in mixture B

MB-HAZ = x_6 = the number of hazel nuts in mixture B

MD-CAS = x_7 = the number of cashews in mixture D

MD-PEA = x_8 = the number of peanuts in mixture D

MD-HAZ = x_9 = the number of hazel nuts in mixture D

A listing of the cards in our deck for this run follows:

```
@ RUN,/R    RCTPWB,NHGRCT,LPPWB,15,50
@FREE TPF$
@ASG,T TPF$,F///500
@ASG,T LPTAPE,T,1815
@ASG,T 9,T,SCRTCH
@FIND,A LPTAPE.LP1108/E8S
@CCPIN,A LPTAPE.LP1108/E8S,TPF$.
@FREE LPTAPE
@XQT .LP1108/E8S
LOAD
ROW ID
     5      PROFIT      +MIX A1      +MIX A2      +MIX B1      +MIX B2
     3     +CAP  C      +CAP  P      +CAP  H
MATRIX
        MA-CASPROFIT-     .15
        MA-CASMIX A1-     .5
        MA-CASMIX A2-     .25
        MA-CASCAP  C     1.0
        MA-PEAPROFIT      .25
        MA-PEAMIX A1      .5
        MA-PEAMIX A2      .75
        MA-PEACAP  P     1.0
        MA-HAZPROFIT      .15
        MA-HAZMIX A1      .50
        MA-HAZMIX A2-     .25
```

```
        MA-HAZCAP   H     1.0
        MB-CASPROFIT-      .30
        MB-CASMIX R1-      .75
        MB-CASMIX R2-      .50
        MB-CASCAP   C     1.0
        MB-PEAPROFIT       .1
        MB-PEAMIX R1       .25
        MB-PEAMIX R2       .5
        MB-PEACAP   P     1.0
        MB-HAZMIX R1       .25
        MB-HAZMIX R2-      .5
        MB-HAZCAP   H     1.0
        MD-CASPROFIT-      .4
        MD-CASCAP   C     1.0
        MD-PEACAP   P     1.0
        MD-HAZPROFIT-     1.0
        MD-HAZCAP   H     1.0
FIRSTB
               CAP  C   100.
               CAP  P   100.
               CAP  H   60.
ENDATA
MATRIX
EQLIST
MTXMAP
GOGOGO
PRIMAL
DUAL
REDCST
RANGES
ERRORS
ENDJOB
& FIN
```

As a result, we obtain the information given on the following pages.

CASE

MATRIX TABLEAU

LABEL	COST	B-VEC	MA-CAS	MA-PEA	MA-HAZ	MB-CAS	MB-PEA	MB-HAZ	MU-CAS
COST	.000000	.000000	-.150000	.250000	.150000	-.300000	.100000	.000000	-.400000
NO. LABEL									
1 PROFIT	.000000	.000000	-.150000	.250000	.150000	-.300000	.100000	.000000	-.400000
2 MIX A1	.000000	.000000	-.500000	.500000	.500000	.000000	.000000	.000000	-.400000
3 MIX A2	.000000	.000000	-.250000	.750000	-.250000	.000000	.000000	.000000	.000000
4 MIX B1	.000000	.000000	.000000	.000000	.000000	-.750000	.250000	.250000	.000000
5 MIX B2	.000000	.000000	.000000	.000000	.000000	-.500000	.500000	-.500000	.000000
6 CAP C	.000000	100.000000	1.000000	.000000	.000000	1.000000	.000000	.000000	.000000
7 CAP P	.000000	100.000000	.000000	1.000000	.000000	.000000	1.000000	.000000	1.000000
8 CAP H	.000000	60.000000	.000000	.000000	1.000000	.000000	.000000	1.000000	.000000

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```
20 AUG 70                                                                                    .001.007.001

CASE                      ENLIST OUTPUT

EQUATION    1 LABEL PROFIT COST =     .000000 ORIGINAL BI =     .000000
+   1.00000(PROFIT) -    .15000(MA-CAS) +    .25000(MA-PEA) +    .15000(MA-MAZ) -    .30000(MB-CAS) +    .10000(MB-PEA)
-    .40000(MD-CAS) -   1.00000(MD-MAZ)

EQUATION    2 LABEL MIX A1 COST =     .000000 ORIGINAL BI =     .000000
+   1.00000(MIX A1) -    .50000(MA-CAS) +    .50000(MA-PEA) +    .50000(MA-MAZ)

EQUATION    3 LABEL MIX A2 COST =     .000000 ORIGINAL BI =     .000000
+   1.00000(MIX A2) -    .25000(MA-CAS) +    .75000(MA-PEA) -    .25000(MA-MAZ)

EQUATION    4 LABEL MIX B1 COST =     .000000 ORIGINAL BI =     .000000
+   1.00000(MIX B1) -    .75000(MB-CAS) +    .25000(MB-PEA) +    .25000(MB-MAZ)

EQUATION    5 LABEL MIX B2 COST =     .000000 ORIGINAL BI =     .000000
+   1.00000(MIX B2) -    .50000(MB-CAS) +    .50000(MB-PEA) -    .50000(MB-MAZ)

EQUATION    6 LABEL CAP  C COST =     .000000 ORIGINAL BI =  100.000000
+   1.00000(CAP  C) +   1.00000(MA-CAS) +   1.00000(MB-CAS) +   1.00000(MD-CAS) + 100.00000(*BI   )

EQUATION    7 LABEL CAP  P COST =     .000000 ORIGINAL BI =  100.000000
+   1.00000(CAP  P) +   1.00000(MA-PEA) +   1.00000(MB-PEA) +   1.00000(MD-PEA) + 100.00000(*BI   )

EQUATION    8 LABEL CAP  H COST =     .000000 ORIGINAL BI =   60.000000
+   1.00000(CAP  H) +   1.00000(MA-MAZ) +   1.00000(MB-MAZ) +   1.00000(MD-MAZ) +  60.00000(*BI   )

END ENLIST OUTPUT
```

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CASE MATRIX TABLEAU

LABEL	MU-PEA	MU-HAZ	B1
COST	.000000	-1.000000	.000000
ROW LABEL			
1 PROFIT	.000000	-1.000000	.000000
2 MIX A1	.000000	.000000	.000000
3 MIX A2	.000000	.000000	.000000
4 MIX B1	.000000	.000000	.000000
5 MIX B2	.000000	.000000	.000000
6 CAP C	.000000	.000000	100.000000
7 CAP P	1.000000	.000000	100.000000
8 CAP H	.000000	1.000000	40.000000

END OF MATRIX TABLEAU

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	CUST	B-VE	MA-CCS	MA-PAA	MA-HEZ	MB-CAS	MB-PAA	MB-HEZ	MD-CAS	MD-PAA	MD-HEZ	.81
COST	.	.	-5	5	5	-5	4	.	-5	.	-1	.
PROFIT	.	.	-5	5	5	-5	4	.	-5	.	-1	.
MIX A1	.	.	-5	5	5	.	.	.	.	.	.	.
MIX A2	.	.	-5	5	-5	.	.	.	.	.	.	.
MIX B1	.	.	.	.	.	-5	5	5	.	.	.	.
MIX B2	.	.	.	.	.	-5	5	-5	.	.	.	.
CAP C	.	7	1	.	.	1	.	.	1	.	.	7
CAP P	.	7	.	1	.	.	1	.	.	1	.	7
CAP H	.	7	.	.	1	.	.	1	.	.	1	7

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THE MATRIX ELEMENTS ARE REPRESENTED AS FOLLOWS

CODE	MAGNITUDE OF ELEMENTS GREATER EQUAL TO OR THAN	LESS THAN	OCCURENCES
0	0.0	0.0001	0
2	0.0001	0.001	0
3	0.001	0.01	0
4	0.01	0.1	2
5	0.1	0.99999999	22
1	=1.0		11
6	1.0	10.0	0
7	10.0	100.0	6
8	100.0	1000.0	0
9	1000.0		0

END MTXMAP OUTPUT

20 AUG 70 .001.014.001

PRIMAL OUTPUT

CASE ITERATION 5 OBJECTIVE VALUE 5.0000000

	LABEL	COST	ACTIVITY	LABEL	COST	ACTIVITY	LABEL	COST	ACTIVITY
E	PROFIT	.000000	-.500000D+01	CAP P	.000000	50.000000	CAP H	.000000	10.000000
	MA-CAS	-.150000	100.000000	MA-PEA	.250000	50.000000	MA-MAZ	.150000	50.000000
	MB-PEA	.100000	.000000	MB-MAZ	.000000	.000000			

END PRIMAL OUTPUT

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DUAL OUTPUT

CASE ITERATION 5 OBJECTIVE VALUE 5.0000000

LABEL	COST	SHADOW PRICE	LABEL	COST	SHADOW PRICE	LABEL	COST	SHADOW PRICE
E PROFIT	.000000	.000000	MIX A1	.000000	.340000	MIX A2	.000000	.100000
MIX B1	.000000	.200000	MIX B2	.000000	.100000	CAP C	.000000	.050000
CAP P	.000000	.000000	CAP H	.000000	.000000			

END DUAL OUTPUT

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REDCST OUTPUT

CASE ITERATION 5 OBJECTIVE VALUE 5.0000000

LABEL	COST	REDUCED COST	LABEL	COST	REDUCED COST	LABEL	COST	REDUCED COST
MIX A1	.000000	.350000	MIX A2	.000000	.100000	MIX B1	.000000	.200000
MIX B2	.000000	.100000	CAP C	.000000	.050000	MD-CAS	-.300000	.150000
MU-CAS	-.400000	.450000	MD-PEA	.000000	.000000	MD-HAZ	-1.000000	1.000000
Z .B1	.000000	5.000000						

END REDCST OUTPUT

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DUAL RANGE OUTPUT

CASE ITERATION 5 OBJECTIVE VALUE 5.0000000

		- - - - - - LIMITS OF RANGE		- - - - - -	
LABEL	ORIG. ACT.	LABEL	INCREMENT	LABEL	INCREMENT
MIX A1	.000000	MA-HAZ	-33.333333	CAP H	6.666667
MIX A2	.000000	CAP H	-10.000000	MA-HAZ	50.000000
MIX B1	.000000	MB-PEA	.000000	CAP H	5.000000
MIX B2	.000000	MB-PEA	.000000	MB-HAZ	.000000
CAP C	100.000000	MA-HAZ	-100.000000	CAP H	20.000000
MU-CAS	.000000	MB-PEA	.000001	CAP H	20.000000
MU-CAS	.000000	CAP H	-20.000000	MA-HAZ	100.000000
MU-PEA	.000000		-9999.000000	CAP P	50.000000
MU-HAZ	.000000		-9999.000000	CAP H	10.000000

END DUAL RANGE OUTPUT

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PRIMAL RANGE OUTPUT

CASE ITERATION 5 OBJECTIVE VALUE 5.0000000

		- - - - - - LIMITS OF RANGE	- - - - - -		
LABEL	COST	LABEL	INCREMENT	LABEL	INCREMENT
PROFIT	.000000		-9999.000000	MIX A2	1.000000
CAP P	.000000	MD-PEA	.000000	MIX B1	.100000
CAP H	.000000	MIX B2	-.100000	MIX B1	.100000
MA-CAS	-.150000	CAP C	-.050000		9999.000000
MA-PEA	.250000	MIX A2	-.100000		9999.000000
MA-HAZ	.150000	CAP C	-.100000	MIX A2	.100000
MB-PEA	.100000	MIX B1	-.100000	MB-CAS	.075000
MB-HAZ	.000000	MIX B1	-.100000	MIX B2	.100000

END PRIMAL RANGE OUTPUT

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DUAL ERROR ANALYSIS.

LABEL	Z(J)	C(J)	ERROR
PROFIT	.000	.000	.00000000
CAP P	.000	.000	.00000000
CAP M	.000	.000	.00000000
LA-CAS	-.150	-.150	.00000000
MA-PEA	.250	.250	.00000000
MA-MAZ	.150	.150	.00000000
MS-PEA	.100	.100	.00000000
MA-MAZ	.000	.000	.00000000

MAXIMUM DUAL ERROR OF .00000000 FOR VARIABLE

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PRIMAL ERROR ANALYSIS .

LABEL	ORIG B(I)	CALC B(I)	ERROR
PROFIT	.000	.000	-.35762787-06
MIX A1	.000	.000	.00000000
MIX A2	.000	.000	.00000000
MIX B1	.000	.000	.00000000
MIX B2	.000	.000	.00000000
CAP C	100.000	100.000	.00000000
CAP P	100.000	100.000	.00000000
CAP M	60.000	60.000	.00000000

MAXIMUM PRIMAL ERROR OF -.35762787-06 FOR VARIABLE PROFIT

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